4.6 Related Rates

Any equation involving two or more variables that are differentiable functions of time, t, can be used to find an equation that relates their corresponding rates.

For example - Suppose that a particle P(x, y) is moving along a curve C in the plane so that its coordinates x and y are differentiable functions of time t. If D is the distance from the origin to P, then using the Chain Rule we can find an equation that relates dD/dt, dx/dt, and dy/dt.

$$D = \sqrt{x^2 + y^2}$$

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}}(2x\frac{dx}{dt} + 2y\frac{dy}{dt})$$

What does dD/dt represent? dx/dt? dy/dt?

Example 1:

Assume that the radius, r, of a sphere is a differentiable function of t and let V be the volume of the sphere. Find an equation that relates dV/dt and dr/dt.

$$V = \frac{4}{3}\pi r^3 \leftarrow \text{Volume of sphere}$$

$$\frac{dV}{dt} = 3\left(\frac{4}{3}\pi\right) r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Example 2:

a. Assume that the radius, r, and height, h, of a cone are differentiable functions of t and let V be the volume of the cone. Find an equation that relates dV/dt, dr/dt, and dh/dt.

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (r^2 h)$$

$$\frac{dV}{dt} = \frac{1}{3}\pi \left(r^2 \cdot \frac{dh}{dt} + h \cdot 2r \frac{dr}{dt}\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left(r^2 \frac{dh}{dt} + 2rh \frac{dr}{dt}\right)$$

b. If the volume is increasing at a rate of 2 cm^3/sec and the height is decreasing at a rate of 1/3 cm/sec, what is the value of dV/dt and dh/dt?

$$\frac{dV}{dt} = 2 \qquad \frac{dh}{dt} = -\frac{1}{3}$$

Strategy for Solving Related Rate Problems:

1. Understand the Problem
2. Develop a mathematical model for the problem - draw a diagram and label it to help. Also, distinguish between quantities that are constant and those that will change over time.
3. Write an equation relating the variable whose rate of change you seek with the variable(s) whose rate of change you know.
4. Differentiate both sides of the equation implicitly with respect to time, t.
5. Substitute values for any quantities that depend on time - Do this AFTER you differentiate!!! If you substitute before, you "freeze the picture" and make changeable values behave like constants with zero derivatives.
6. Interpret the solution.

Homework:
p. 251 #1-7

$\frac{dV}{dt} = 100\pi \text{ ft}^3/\text{min}$ $\frac{dr}{dt} = ?$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$100\pi = 4\pi(5)^2 \frac{dr}{dt}$

$100\pi = 100\pi \frac{dr}{dt}$

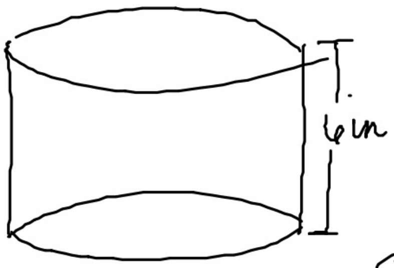
$\boxed{\frac{dr}{dt} = 1 \text{ ft/min}}$

$S = 4\pi r^2$

$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

$\frac{dS}{dt} = 8\pi(5)(1)$

$\boxed{\frac{dS}{dt} = 40\pi \text{ ft}^2/\text{min}}$



$$\frac{dr}{dt} = \frac{.001}{3} = \frac{1}{3000} \text{ in/min} \qquad \frac{dV}{dt} = ?$$

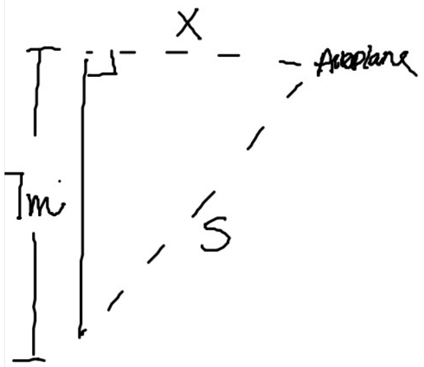
$$V = \pi r^2 h = \pi r^2 (6)$$

Eq: $V = 6\pi r^2$

$$\frac{dV}{dt} = 12\pi r \frac{dr}{dt}$$

$$\frac{dV}{dt} = 12\pi(1.9)\left(\frac{1}{3000}\right)$$

$$\boxed{\frac{dV}{dt} = .0239 \text{ in}^3/\text{min}}$$



$$\frac{ds}{dt} = 300\text{mph} \qquad \frac{dx}{dt} = ?$$

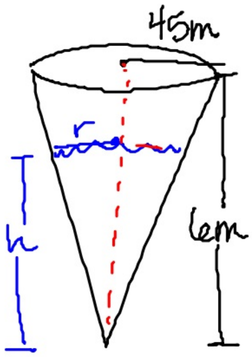
$$7^2 + x^2 = s^2$$

$$x = \sqrt{s^2 - 49}$$

$$\frac{dx}{dt} = \frac{1}{2}(s^2 - 49)^{-\frac{1}{2}} \cdot 2s\frac{ds}{dt}$$

$$\frac{dx}{dt} = \frac{\cancel{2}s\frac{ds}{dt}}{\cancel{2}\sqrt{s^2-49}} = \frac{s\frac{ds}{dt}}{\sqrt{s^2-49}} = \frac{10(300)}{\sqrt{51}}$$

$$\boxed{\approx 420.08\text{mph}}$$



$$\frac{dV}{dt} = -50\,m^3/min$$

$$\frac{dh}{dt} = ?$$

Similar △'s

$$\frac{6}{45} = \frac{h}{r}$$

$$\boxed{r = 7.5h}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi (7.5h)^2 h$$

$$V = 18.75\pi h^3$$

$$\frac{dV}{dt} = 56.25\pi h^2 \frac{dh}{dt}$$

$$-50 = 56.25\pi (5)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = -.0113\,m/min$$

$$\frac{dh}{dt} = -1.13\,cm/min$$

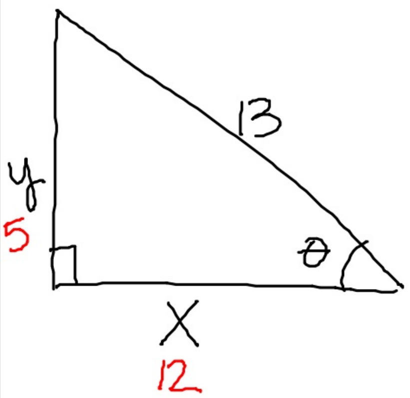
Falling @ a rate of 1.13 cm/min.

$$r = 7.5h$$

$$\frac{dr}{dt} = 7.5\frac{dh}{dt} = 7.5(-1.13)$$

$$\frac{dr}{dt} = -8.49 \text{ cm/min}$$

Changing at a rate of -8.49 cm/min



$$\frac{dx}{dt} = 5\,\text{ft/sec} \qquad \frac{dy}{dt} = ?$$

$$\theta = \tan^{-1}\left(\frac{5}{12}\right)$$

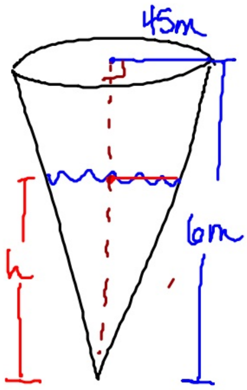
$$\theta = .3948\,\text{rad}$$

$$\tan\theta = \frac{y}{X}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{X\frac{dy}{dt} - y\frac{dx}{dt}}{X^2}$$

$$\sec^2(.3948)\frac{d\theta}{dt} = \frac{12(-12) - 5(5)}{12^2}$$

$$\frac{d\theta}{dt} = -1\,\text{rad/sec}$$



$$\frac{dV}{dt} = -50\,m^3/min \qquad \frac{dh}{dt} = ?$$

Similar Δ's

$$\frac{6}{45} = \frac{h}{r}$$

$$r = 7.5h$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi(7.5h)^2 h$$

$$\frac{dh}{dt} = -1.13\,cm/min$$

Eq: $V = 18.75\pi h^3$

$$\frac{dV}{dt} = 56.25\pi h^2 \frac{dh}{dt}$$

$$-50 = 56.25\pi(5)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = -.0113\,m/min$$

$$r = 7.5h$$

$$\frac{dr}{dt} = 7.5\frac{dh}{dt}$$

$$\frac{dr}{dt} = 7.5(-1.13) = -8.49 \text{ cm/min}$$

$$\frac{dV}{dt} = 100\pi \text{ ft}^3/\text{min}$$

a.) $\frac{dr}{dt} = ?$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$100\pi = 4\pi(5)^2 \frac{dr}{dt}$$

$$100\pi = 100\pi \frac{dr}{dt}$$

$$\frac{dr}{dt} = 1 \text{ ft/min}$$

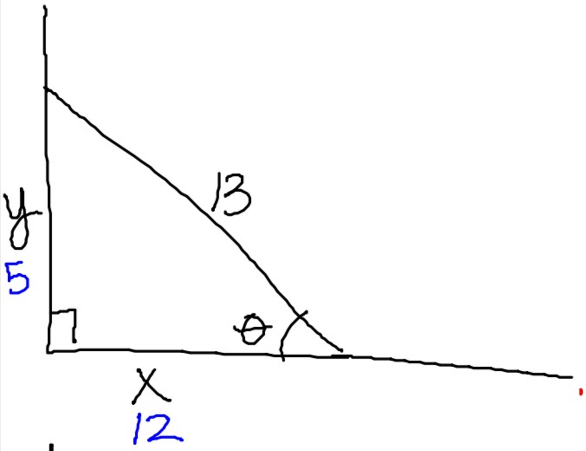
b.) $\frac{dS}{dt} = ?$

$$S = 4\pi r^2$$

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

$$\frac{dS}{dt} = 8\pi(5)(1)$$

$$\frac{dS}{dt} = 40\pi \text{ ft}^2/\text{min}$$



$$\frac{dx}{dt} = 5 \text{ ft/sec}$$

$$\theta = \tan^{-1}\left(\frac{5}{12}\right)$$

$$\tan\theta = \frac{y}{x}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$$

$$\sec^2(.3948)\frac{d\theta}{dt} = \frac{12(-12) - 5(5)}{12^2}$$

$$\frac{d\theta}{dt} = -1 \text{ rad/sec}$$