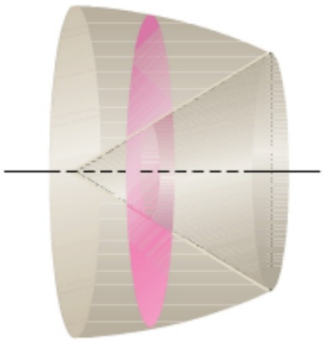
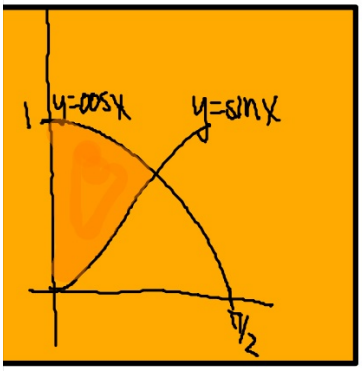
EXAMPLE 3 Washer Cross Sections

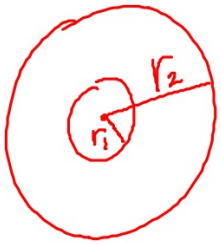
The region in the first quadrant enclosed by the y-axis and the graphs of $y = \cos x$ and $y = \sin x$ is revolved about the x-axis to form a solid. Find its volume.





Cross-sections washers:

$$A = \pi(r_2^2 - r_1^2)$$



$$A(x) = \pi(\cos^2 x - \sin^2 x)$$

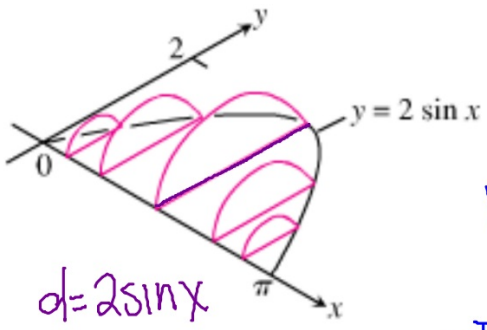
$$V = \int_0^{\frac{\pi}{4}} \pi(\cos^2 x - \sin^2 x)\,dx = \pi\int_0^{\pi/4} \cos 2x\,dx$$

$$= \pi\left[\frac{1}{2}\sin 2x\right]_0^{\pi/4} = \pi\left(\frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{2}\sin 0\right)$$

$$= \pi\left(\frac{1}{2} - 0\right) = \boxed{\frac{\pi}{2}\text{ units}^3}$$

EXAMPLE 6 A Mathematician's Paperweight

A mathematician has a paperweight made so that its base is the shape of the region between the x-axis and one arch of the curve $y = 2 \sin x$ (linear units in inches). Each cross section cut perpendicular to the x-axis (and hence to the xy-plane) is a semicircle whose diameter runs from the x-axis to the curve. (Think of the cross section as a semicircular fin sticking up out of the plane.) Find the volume of the paperweight.



Cross-sections: semicircles $\left(\frac{1}{2}\pi r^2\right)$

$d = 2\sin x$

$r = \frac{2\sin x}{2} = \sin x$

$A(x) = \frac{1}{2}\pi \sin^2 x$

$$V = \int_0^{\pi} \frac{\pi}{2}\sin^2 x\,dx = \frac{\pi}{2}\int_0^{\pi} \sin^2 x\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx = \frac{\pi}{4}\int_0^{\pi}(1 - \cos 2x)\,dx$$

$$= \frac{\pi}{4}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi}{4}\left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right]$$

$$= \frac{\pi}{4}(\pi - 0) = \frac{\pi^2}{4} \approx 2.47 \text{ in}^3$$

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