Jackson

AP Calculus AB

AP Test Review

Formulas and Theorems

1. Limits and Continuity

A function $y = f(x)$ is continuous at $x = a$ if:

1. $f(a)$ is defined (exists)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

Otherwise, f is discontinuous at $x = a$.

The limit $\lim_{x \to a} f(x)$ exists if and only if both corresponding one-sided limits exist and are equal.

$$\lim_{x \to a} f(x) = L \quad \Leftrightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

2. Intermediate Value Theorem

A function $y = f(x)$ that is continuous on a closed interval $[a,b]$ takes on every value between $f(a)$ and $f(b)$.

If f is continuous and $[a,b]$ and $f(a)$ and $f(b)$ differ in sign, then the equation $f(x) = 0$ has at least one solution in the open interval (a,b).

3. Limits of Rational Functions as $x \to \pm\infty$

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1. $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = 0$ if the degree of $f(x)$, the degree of $g(x)$

2. $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is infinite of the degree of $f(x) >$ the degree of $g(x)$

3. $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ if the degree of $f(x)$ = the degree of $g(x)$ The limit will be the ratio of the leading coefficient of $f(x)$ to $g(x)$.

4. <u>Average and Instantaneous Rate of Change</u>

1. Average Rate of Change = $\frac{f(b)-f(a)}{b-a}$

2. Instantaneous Rate of Change = $f'(x)$

5. <u>Rolle's Theorem</u>

If f is continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) = f(b)$, then there is at least one number c in the open interval such that $f'(c) = 0$

6. <u>Mean Value Theorem</u>

If f is continuous on $[a,b]$ and differentiable on (a,b), then there is at least one number c in (a,b) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$

7. <u>Extreme Value Theorem</u>

If f is continuous on a closed interval $[a,b]$, then $f(x)$ has both an absolute maximum and an absolute minimum on $[a,b]$.

8. To find the maximum and minimum values of a function $y = f(x)$, locate

   1. the points where $f'(x) = 0$ or where $f'(x)$ fails to exist
   2. the end points, if any on the domain of $f(x)$

Note: These are the only candidates for the value of x where $f(x)$ may have a maximum or a minimum.

9. Let f be differentiable for $a < x < b$ and continuous for $a \leq x \leq b$.

   1. If $f'(x) > 0$ for every x in (a,b), then f is increasing on $[a,b]$.
   2. If $f'(x) < 0$ for every x in (a,b), then f is decreasing on $[a.b]$.

10. Suppose that $f''(x)$ exists on the interval (a,b).

1. If $f''(x) > 0$ in (a,b), then f is concave upward in (a,b).
2. If $f''(x) < 0$ in (a,b), then f is concave downward in (a,b).

To locate the points of inflection of $y = f(x)$, find the points where $f''(x) = 0$ or where $f''(x)$ fails to exist. These are the only candidates where $f(x)$ may have a point of inflection. Then test these points to make sure that $f''(x) < 0$ on one side and $f''(x) > 0$ on the other.

11. If a function is differentiable at a point $x = a$, it is continuous at that point. The converse is false, i.e. continuity does not imply differentiability.

12. Inverse Functions

1. If f and g are two functions such that $f(g(x) = x$ for every x in the domain of g, and $g(f(x)) = x$ for every x in the domain of f, then f and g are inverse functions of each other.
2. A function f has an inverse function if and only if no horizontal line intersects its graph more than once.
3. If f is either increasing or decreasing in an interval, then f has an inverse function over that interval.
4. If f is differentiable at every point on an interval I, and $f'(x) \neq 0$ on I, then

   $g = f^{-1}(x)$ is differentiable at every point of the interior of the interval $f(I)$ and

$$g'(f(x)) = \frac{1}{f'(x)}$$

13. Properties of $y = e^x$

1. The exponential function $y = e^x$ is the inverse function of $y = \ln x$.
2. The domain is the set of all real numbers.
3. The range is the set of all positive numbers.
4. $\frac{d}{dx}(e^x) = e^x$
5. $e^a \bullet e^b = e^{a+b}$
6. $y = e^x$ is continuous, increasing and concave up.

14. Properties of $y = \ln x$

1. The domain of $y = \ln x$ is the set of all positive numbers.
2. The range of $y = \ln x$ is the set of all real numbers.
3. $y = \ln x$ is continuous and increasing everywhere on its domain.
4. $\ln(ab) = \ln a + \ln b$
5. $\ln(\frac{a}{b}) = \ln a - \ln b$
6. $\ln a^r = r \ln a$

15. Properties of the Definite Integral

Let $f(x)$ and $g(x)$ be continuous on $[a,b]$.

1. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ for any constant c.
2. $\int_a^a f(x)dx = 0$
3. $\int_b^a f(x)dx = -\int_a^b f(x)dx$
4. If $f(x)$ is an odd function, then $\int_{-a}^a f(x)dx = 0$
5. If $f(x)$ is an even function, then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$
6. If $f(x) \geq 0$ on $[a,b]$ then $\int_a^b f(x)dx \geq 0$

16. Fundamental Theorem of Calculus

$\int_a^b f(x)dx = F(b) - F(a)$, where $F'(x) = f(x)$ or $\frac{d}{dx}\int_a^x f(x)dx = f(x)$

17. Position, Velocity, Speed, and Acceleration

1. Total displacement = $\int_a^b v(t)dt$ (How far you are from where you started)
2. Total distance traveled $\int_a^b |v(t)|\,dt$ (How far you've gone)
3. The velocity of an object tells how fast it is going and in which direction.
4. The speed of an object is the absolute value of the velocity.

If velocity and acceleration have the same sign, speed is increasing. If velocity and acceleration have opposite signs, speed is decreasing.

5. Acceleration is the instantaneous rate of change of velocity. It is the derivative of velocity.

6. Average velocity = $\frac{x(b)-x(a)}{b-a}$ = $\frac{1}{b-a}\int_a^b v(t)dt$

If x is the displacement of a moving object and t is time, then

a) velocity = $v(t) = x'(t) = \frac{dx}{dt}$

b) acceleration = $a(t) = x''(t) = v'(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

18. Average Value of f(x) on [a,b] is $\frac{1}{b-a}\int_a^b f(x)dx$

19. Area Between Curves

If f and g are continuous functions such that $f(x) \geq g(x)$ on [a,b], then the area between the curves is $\int_a^b (f(x)-g(x))dx$

20. Volumes of Solids of Revolution

Let f be nonnegative and continuous on [a,b], and let R be the region bounded above by $y = f(x)$, below by the x-axis, and the sides by the lines $x = a$ and $x = b$.

1. When this region R is revolved about the x-axis, it generates a solid whose volume

$$V = \int_a^b \pi[f(x)]^2 dx$$

21. Volumes of Solids with Known Cross Sections

1. For cross sections of area $A(x)$ taken perpendicular to the x-axis volume $= \int_a^b A(x)dx$
2. For cross sections of area $A(x)$ taken perpendicular to the Y-axis volume $= \int_c^d A(y)dy$

22. Solving Differential Equations by Separating the Variables.

There are many techniques for solving differential equations. Any differential equation you may be asked to solve can be solved by separating the variables. Rewrite the equation as an equivalent equation with all the x and dx terms on one side and all the y and dy terms on the other. Integrate both sides to obtain an equation without dx or dy, but with one constant of integration. Use the initial condition to evaluate this constant.

Looking at the Big Picture in Preparing Students for the AP Examinations

By Dixie Ross

How can we find the derivative?

LIMIT DEF'S

1. By the limit definition:

$$\lim_{h\to 0}\frac{\sin(\frac{2\pi}{3}+h)-\sin(\frac{2\pi}{3})}{h}$$

$f(x)=\sin x$
Find $f'(2\pi/3)$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\lim_{x\to \frac{\pi}{3}}\frac{\tan x-\tan(\frac{\pi}{3})}{x-\frac{\pi}{3}}$$

$f(x)=\tan x$
Find $f'(\pi/3)$

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

2. By formulas

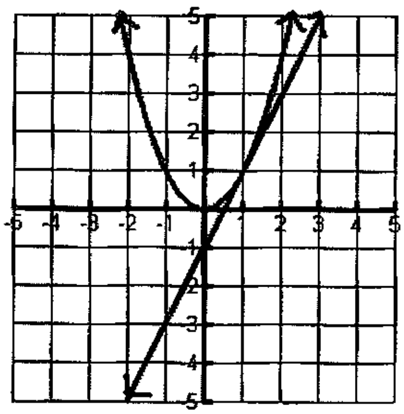
Find F'(x) $\qquad F(x)=x^3\cos(3x)$

$$F'(x)=x^3(-\sin(3x)\cdot 3)+\cos(3x)(3x^2)$$
$$=\boxed{-3x^3\sin(3x)+3x^2\cos(3x)}$$

3. By graphs

Find F'(1) = slope of tangent line @ x=1

$\boxed{F'(1)=2}$



4. By approximation

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
f(x)	6	6.3	6.8	6.4	6.6	6.9	7.0	7.4

Approximate $f'(2.5) = \frac{f(3)-f(2)}{3-2} = \frac{7-6.6}{1} = \boxed{.4}$

5. Implicitly

$x^3 + 2x^2y - 3y^3 = 10$

$3x^2 + 2x^2\frac{dy}{dx} + 4xy - 9y^2\frac{dy}{dx} = 0$

$\frac{dy}{dx}(2x^2 - 9y^2) = -4xy - 3x^2$

$\boxed{\frac{dy}{dx} = \frac{-4xy-3x^2}{2x^2-9y^2}}$

6. By technology

$F(x) = \frac{\sin(2x)\sqrt{x^3+1}}{\csc(x^2)(3x-5)^4}$ find F'(1)

$\boxed{F'(1) \approx 14.196}$ .481

7. Using the Second Fundamental Theorem of Calculus.

$F(x) = \int_3^{x^2} \sqrt{t^4+1}\,dt$ find F'(x) $= \sqrt{(x^2)^4+1} \cdot 2x$

$= \boxed{2x\sqrt{x^8+1}}$

How can we use derivatives?

1. To determine slopes:

Write the equation of a line tangent to $f(x) = x^3 + 4x^2 - 5x + 1$ at x = -1. $f'(x) = 3x^2 + 8x - 5$

$m = f'(-1) = 3(-1)^2 + 8(-1) - 5 = 3 - 8 - 5 = -10$

$f(-1) = (-1)^3 + 4(-1)^2 - 5(-1) + 1 = -1 + 4 + 5 + 1 = 9$ $\boxed{y - 9 = -10(x+1)}$

Write the equation of a line normal to the function given above at x = -1.

$\boxed{y - 9 = \frac{1}{10}(x+1)}$

2. To determine where the slopes meet particular conditions.

Find all points on the graph of $x + xy + 2y^2 = 6$ where the $\overbrace{\text{slope of the tangent line}}^{\frac{dy}{dx}} \text{ is } \frac{-1}{3}$. $= -\frac{1}{3}$

$1 + x\frac{dy}{dx} + y + 4y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-1-y}{x+4y}$

$\frac{-1-y}{x+4y} = -\frac{1}{3}$

$-3-3y = -x-4y$

$\boxed{3+3y = x+4y}$

3. To determine rates of change

Find the rate of change in the volume of a sphere when the radius is 3 ft and expanding at a rate of 2 feet per second. $V = \frac{4}{3}\pi r^3$

Want $\frac{dV}{dt}$

when $r=3$

$\frac{dr}{dt} = 2$

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2\frac{dr}{dt}$

$dV/dt = 4\pi r^2\frac{dr}{dt} = 4\pi(3)^2(2) = \boxed{72\pi \text{ ft}^3/\text{sec}}$

If the temperature of water is given by $T = 6e^{(-.3t)} + 60$, how fast is the temperature changing when t = 10?

4. To ascertain information about movement

If the position of a particle moving on the y-axis is given by $y(t) = t^3 - 7t^2 + 15t + 4$ for $t \geq 0$

position ↓

$v(t) = 3t^2 - 14t + 15$

a) When is the particle moving down? $v(t) < 0$

b) When the particle first changes direction, is the velocity increasing?

c) When is the instantaneous rate of change equal to the average rate of change for $0 \leq t \leq 2$

d) What is the total distance traveled for $2 \leq t \leq 4$?

5. To determine intervals on which a function is increasing/decreasing

$$F(x)=\frac{x^2-2x+4}{x-2}$$

6. To determine intervals of concavity and points of inflection

$$F(x)=x^4-12x^3+48x^2-64x$$

7. As a way of determining absolute extrema

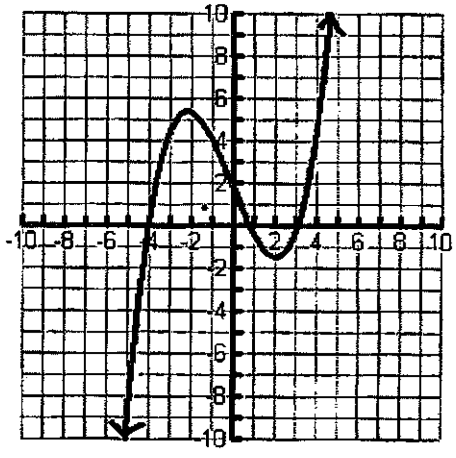
Find the maximum and minimum values of $F(x)=x^3-3x^2+12$ on the closed interval [-1,4].

8. As a way of determining local extrema

If $h'(x)=\frac{x^2-9}{x-2}$ for all $x\neq 2$, find all critical points for h(x) and determine whether these points are local maximum, local minimum, or neither.

9. To determine the features of a function

Given F(-5) = 3, use the graph of F'(x) to sketch a graph of F(x).



How can we find integrals?

1. Using anti-derivatives

$\int(8x^2 - e^{4x} + \sec^2 x)dx$

$$\boxed{\frac{8x^3}{3} - \frac{1}{4}e^{4x} + \tan x + C}$$

~~2.~~ Using the limit definition

$\int_1^4 (2x+5)dx$

3. Geometrically $\sqrt{16-x^2} \rightarrow$ semi-circle (center @ origin, radius 4)

$A = \frac{\pi r^2}{2}$

$\int_{-4}^{0} \sqrt{16-x^2}dx = \frac{1}{2}\left(\frac{1}{2}\pi(4)^2\right) = \boxed{4\pi}$

4. Numerically (using right, left, trapezoid and midpoint methods)

x	0	3	6	7	8	10	12
F(x)	5	7	8	10	9	4	7

Approximate the value of $\int_0^{12} f(x)dx$ using the midpoint method with 3 subintervals.

RRAM: $6(8)+2(9)+4(7) = 94$

LRAM: $6(5)+2(8)+4(9)=82$

MRAM: $6(7)+2(10)+4(4)=78$

Trap: $\frac{6}{2}(5+8)+\frac{2}{2}(8+9)+\frac{4}{2}(9+7)=88$

5. Using the Fundamental Theorem of Calculus

$\int_a^b f(x)dx = F(b) - F(a)$

$\int_{-1}^{3}(6x^2+4x-3)dx$

$\left[\frac{6x^3}{3}+\frac{4x^2}{2}-3x\right]_{-1}^{3} = \left[2x^3+2x^2-3x\right]_{-1}^{3}$

$= \left[2(27)+2(9)-9\right] - \left[2(-1)+2(1)+3\right]$

$= \boxed{60}$

6. Using technology

$\int_1^4 x^2 \ln x\,dx$ Math-9

$\boxed{\approx 22.574}$

<u>How can we use integrals?</u>

1. To find total change

The rate at which water is leaking from a tank is given by $\frac{dV}{dt} = \sqrt{t+1}$ gal/min. Determine how much water has leaked from the tank during the first 8 minutes.

2. Given initial condition, to find final condition

A population of bacteria is growing at a rate of $\frac{dP}{dt} = 2e^{.1t}$ bacteria/hour. If the population after five hours is 2000, find the population after 10 hours.

3. To calculate area

Find the area of the region bounded by $f(x) = \sin x$, $g(x) = \cos x$ and the x-axis.

4. To calculate volumes

a) Using the region described above, find the volume of the solid formed by rotating the region
around $y = 1$.

b) Using the region described above, find the volume of the solid formed by rotating the region
around the y-axis.

c) Using the region described above, find the volume of the solid formed by semi-circular cross sections perpendicular to the x-axis.

5. To ascertain information about motion

1983 AB1 A particle moves along the x-axis so that, at any time $t \geq 0$, its acceleration is given by $a(t) = 6t + 6$. At time $t = 0$, the velocity of the particle is -9 and its position is -27.

a) Find $v(t)$, the velocity of the particle at any time $t \geq 0$.

b) For what values of $t \geq 0$ is the particle moving to the right?

c) Find $x(t)$, the position of the particle at any time $t \geq 0$.

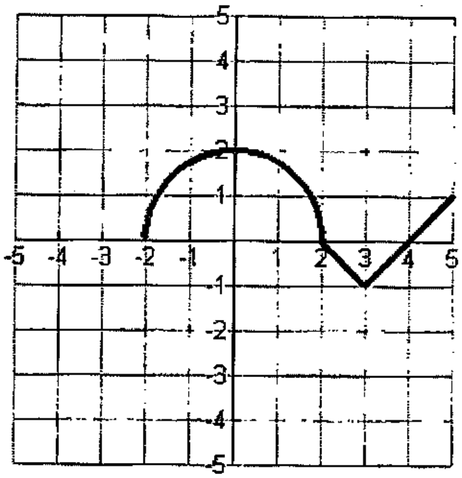
6. To solve differential equations

Find $y = f(x)$ given $\frac{dy}{dx} = 3x^2y$ and $F(2) = e$.

7. As a way of defining functions

The graph of a function f consists of a semicircle and two line segments as shown at the right.
Let g be the function

$$g(x) = \int_0^x f(t)dt$$



a) Find g(4)

b) Find all values of x on the open interval (-2,5) at which g has a relative maximum.

c) Find the x-coordinate of each point of inflection on the graph of g.

LIMITS,

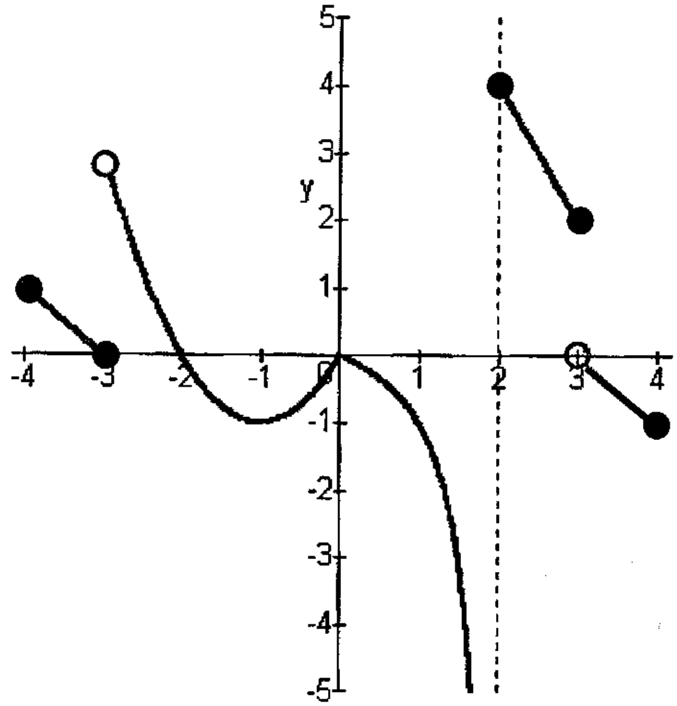
CONTINUITY,

FUNCTIONS

It's Your Turn Problems

I. Functions, Graphs, and Limits

1. Here's the graph of the function f on the interval $[-4,4]$. It has a vertical asymptote at $x=2$, $\lim_{x\to 2^-} f(x) = -\infty$.



a) What are the critical numbers of f?

$x=-3,-1,0,2,3$

b) What is the absolute maximum of f on $[-4,4]$?

4

c) What is the absolute minimum of f on $[-4,4]$?

none

d) Where does f have local maxima?

$x=-4,0,2$

e) Where does f have local minima?

$x=-1,-3,3,4$

f) Where does f appear to be concave-up?

$(-3,0)$

g) Where does f appear to be concave- down?

$(0,2)$

h) Where does f have inflection points?

$x=0$

i) Identify the intervals where f is increasing.

$(-1,0)$

j) Identify the intervals where f is decreasing.

$(-4,-3)(-3,-1)(0,2)(2,3)(3,4)$

k) Find the maximum of f on $[2,3]$.

4

l) Find the maximum of f on $[0,2]$.

0

m) Find the minimum of f on $(-3,0)$.

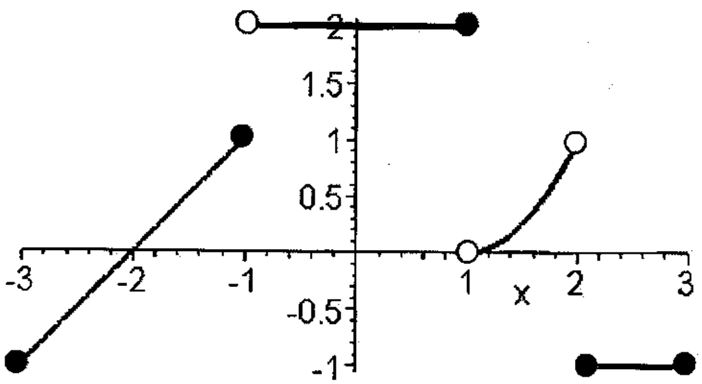
-1

n) Find the maximum of f on $\left(-4,-\frac{5}{2}\right)$.

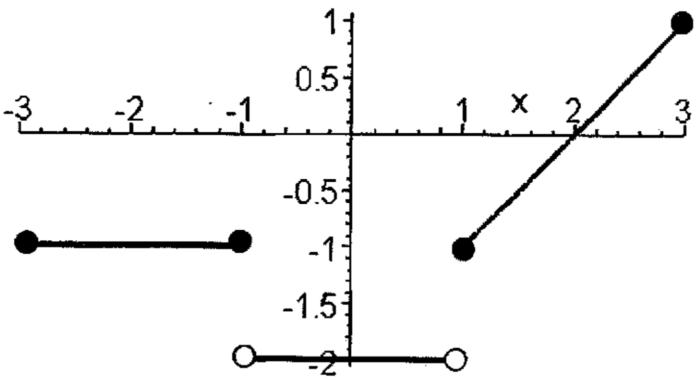
None

2.

Graph of f



Graph of g



a) $\lim_{x\to 0}(f(x)+g(x))$

$2+(-2)$

0

b) $\lim_{x\to -1}(f(x)+g(x))$

DNE

c) $\lim_{x\to 2}(f(x)g(x))$

DNE

d) $\lim_{x\to 2^-}\frac{f(x)}{g(x)}$ $\frac{1}{0}$

e) $\lim_{x\to 2^+}\frac{f(x)}{g(x)}$ $\frac{-1}{0}$

f) $\lim_{x\to 2}\frac{f(x)}{g(x)}$ DNE

g) $\lim_{x\to -1}\frac{f(x)}{g(x)}$ DNE

h) $\lim_{x\to 1^+} f(g(x))$

i) $\lim_{x\to 1^-} f(g(x))$

j) $\lim_{x\to 3^-} f(g(x))$

4. What are the largest and smallest values of the function $f(x)=e^{\cos^2 x}$?

5. What are the largest and smallest values of the function $g(x)=\sin^2 x-4\sin x+5$?

6. What are the largest and smallest values of the function $h(x)=\cos^2 x-\cos x+1$?

7. What are the largest and smallest values of the function $k(x)=\log_3(2+\cos x)$?

8. If $-1<f(x)<-\frac{1}{2}$ for $0<x<1$, and this is all we know about f.

a) Could f be continuous at $x=1$ if $f(1)=0$?

b) Could f be continuous at $x=0$ if $f(0)=0$?

c) Could f be continuous at $x=1$ if $f(1)=-1$?

d) Could f be continuous at $x=0$ if $f(0)=-\frac{3}{4}$?

e) Could f be continuous at $x=\frac{1}{2}$ if $f\left(\frac{1}{2}\right)=-2$?

f) Could f be continuous at $x=\frac{1}{2}$ if $f\left(\frac{1}{2}\right)=-\frac{3}{4}$?

Logarithm Review

Properties of Natural Logs:

a. $\ln 1 = 0$

b. $\ln e = 1$

c. $\ln e^x = x \quad and \quad e^{\ln x} = x$

d. *If* $\ln x = \ln y$, then $x = y$

Simplify using properties of logs:

1. $\ln^{\frac{1}{e}}$

2. $e^{\ln 5}$

3. $\frac{\ln 1}{3}$

4.. $2\ln e$

Change-of-Base Formula
Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$, then $\log_a x$ can be converted to a different base as follows.
Base b: $\log_a x = \frac{\log_b x}{\log_b a}$ — Base 10: $\log_a x = \frac{\log x}{\log a}$ — Base e: $\log_a x = \frac{\ln x}{\ln a}$

Evaluate with a calculator using the Change-of-base formula.

5. $\log_4 25$

6. $\log_2 12$

Properties of Logarithms
Let a be a positive number such that $a \neq 1$, and let n be a real number. If a and v are positive real numbers, the following properties are true.

		Base a	Natural Log
1.	Product property:	$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$
2.	Quotient property:	$\log_a \frac{u}{v} = \log_a u - \log_a v$	$\ln\frac{u}{v} = \ln u - \ln v$
3.	Power property:	$\log_a u^n = n\log_a u$	$\ln u^n = n\ln u$

Using properties of Logarithms

Find the exact value of each expression without using a calculator.

7. $\log_5 \sqrt[3]{5}$

8. $\ln e^6 - \ln e^2$

Expand each logarithmic expression.

9. $\log_4 5x^3y$

10. $\ln\frac{\sqrt{3x-5}}{7}$

11. $\ln\sqrt{\frac{x^2}{y^3}}$

Condense each logarithmic expression.

12. $\frac{1}{2}\log x + 3\log(x+1)$

13. $2\ln(x+2) - \ln x$

14. $\frac{1}{3}[\log_2 x + \log_2(x+1)]$

15. $4\left[\ln z + \ln(z+5)\right] - 2\ln(z-5)$

w

LOG REVIEW

Evaluate each expression. Show work!

1.) $\log_3 27 =$ _____ **2.)** $\ln e^7 =$ _____ **3.)** $\log 1000 =$ _____ **4.)** $\log_6 6^4 =$ _____

5.) $\ln 1 =$ _____ **6.)** $\log_4 64 =$ _____ **7.)** $7^{\log_7 5} =$ _____ **8.)** $\log_3\left(\frac{1}{9}\right) =$ _____

9.) $2^{\log_2 15} =$ _____ **10.)** $\ln\left(\frac{1}{e^5}\right) =$ _____ **11.)** $\log_{\frac{1}{27}} 81 =$ _____ **12.)** $\log_8\left(\frac{1}{2}\right) =$ _____

Use the properties of logarithms to condense each expression. Simplify your answers. Show all steps!

13.) $\log 25 + \log 4 =$ _____ **14.)** $4\ln x - 3\ln y + 5\ln z =$ _____ **15.)** $\frac{1}{2}\log 225 - \frac{1}{3}\log 27 =$ _____

Use the properties of logarithms to expand each expression. Simplify your answers. Show all steps!

16.) $\log_4 \sqrt{j^5 k^7}$ **17.)** $\log \frac{2x^2 y}{3z^4} =$

Solve each equation. Show all work and check your answers!

18.) $\log_6 18 + \log_6(x-2) = 2$ **19.)** $\log(x-3) - \log(x+1) = 1$

20.) $\ln 5 - \ln x = \ln(x-4)$ **21.)** $\log(x-1) + \log(x+2) = \log_{12} 12$

22.) $\log 2x + \log 2x = \log 16$ **23)** $\log 5 + \log x = 2$

Limit Practice

1. Given that $\lim_{x\to a} f(x) = -3$, $\lim_{x\to a} g(x) = 0$, $\lim_{x\to a} h(x) = 8$, find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x\to a}\left[f(x)+h(x)\right] =$ $-3+8$ $\boxed{5}$

(b) $\lim_{x\to a}\left[f(x)\right]^2 =$ $(-3)^2 = \boxed{9}$

(c) $\lim_{x\to a}\sqrt[3]{h(x)} = \sqrt[3]{8} = \boxed{2}$

(d) $\lim_{x\to a}\dfrac{1}{f(x)} = \boxed{-\dfrac{1}{3}}$

(e) $\lim_{x\to a}\dfrac{f(x)}{h(x)} = \boxed{\dfrac{-3}{8}}$

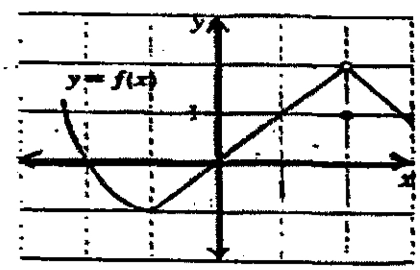
(f) $\lim_{x\to a}\dfrac{g(x)}{f(x)} = \boxed{0}$

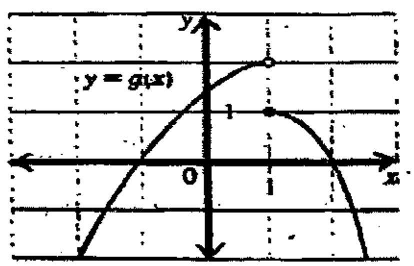
(g) $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{-3}{0}$ $\boxed{\text{DNE}}$

(h) $\lim_{x\to a}\dfrac{2f(x)}{h(x)-f(x)} = \dfrac{2(-3)}{8+3} = \boxed{\dfrac{-6}{11}}$

---

2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.





(a) $\lim_{x\to 2}\left[f(x)+g(x)\right] = 2+0 = \boxed{2}$

(b) $\lim_{x\to 1}\left[f(x)+g(x)\right] = 1 + \text{DNE}$ $\boxed{\text{DNE}}$

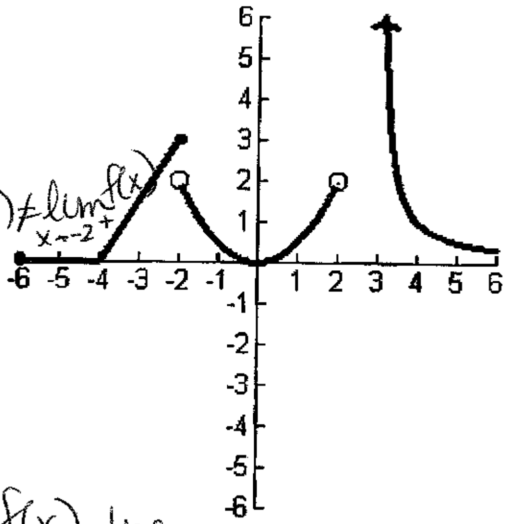
(c) $\lim_{x\to 0}\left[f(x)g(x)\right] = 0\cdot(1.5) = \boxed{0}$

(d) $\lim_{x\to -1}\dfrac{f(x)}{g(x)} = \dfrac{-1}{0}$ $\boxed{\text{DNE}}$

(e) $\lim_{x\to 2} x^3 f(x) = 2^3\cdot 2 = \boxed{16}$

(f) $\lim_{x\to 1}\sqrt{3+f(x)} = \boxed{2}$

True or False. Given the function f(x)



1. $\lim_{x\to -2^-} f(x) = 3$ True

2. $\lim_{x\to -2^+} f(x) = 2$ True

3. $\lim_{x\to -2} f(x) = 2$ False: $\lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x)$

4. $f(-2) = 3$ True

5. $\lim_{x\to -6^+} f(x) = 0$ True

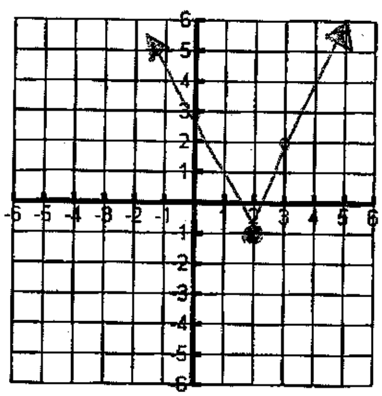
6. $\lim_{x\to -6} f(x) = 0$ False: $\lim_{x\to -6^-} f(x)$ dne

7. $\lim_{x\to 2^-} f(x) = 2$ True

8. $\lim_{x\to 3} f(x) = \text{DNE}$ True

Graph the piecewise functions and answer the following questions:

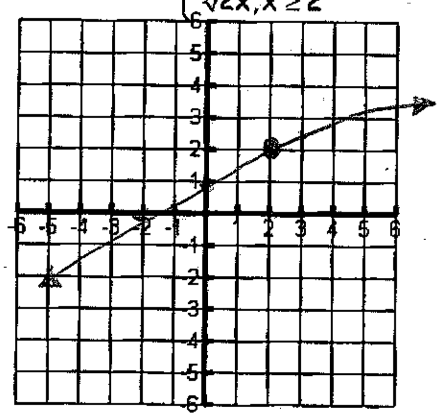
3. $f(x) = \begin{cases} 3-2x, & x<2 \\ 3x-7, & x\geq 2 \end{cases}$



a) $\lim_{x\to 2^+} f(x)$ -1 b) $\lim_{x\to 2^-} f(x)$ -1

c) $\lim_{x\to 2} f(x)$ -1 d) $f(2)$ -1

4. $f(x) = \begin{cases} \frac{1}{2}x+1, & x<2 \\ \sqrt{2x}, & x\geq 2 \end{cases}$



a) $\lim_{x\to 2^+} f(x)$ 2 b) $\lim_{x\to 2^-} f(x)$ 2

c) $\lim_{x\to 2} f(x)$ 2 d) $f(2)$ 2

1) $\lim\limits_{x\to\infty} \dfrac{x+3}{x^2-5x+2}$ $\boxed{0}$

2) $\lim\limits_{x\to-\infty} \dfrac{2x-1}{7-5x}$ $\boxed{-2/5}$

3) $\lim\limits_{x\to\infty} x-\sqrt{x^2+7}$ Numerically $\boxed{0}$

4) $\lim\limits_{x\to 2} x+3$ $\boxed{5}$

5) $\lim\limits_{x\to 13} \sqrt{x+3}$ $\boxed{4}$

6) $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-4}$ $\boxed{\frac{1}{4}}$

7) $\lim\limits_{x\to 2} \dfrac{x-2}{2-\sqrt{x+2}}$ L'H $\cdot \cancel{\dfrac{2+\sqrt{x+2}}{2+\sqrt{x+2}}}$ $\boxed{-4}$

8) $\lim\limits_{x\to 0} \dfrac{\sin x}{3x}$ $\boxed{\frac{1}{3}}$

9) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x}$ L'H $\boxed{0}$

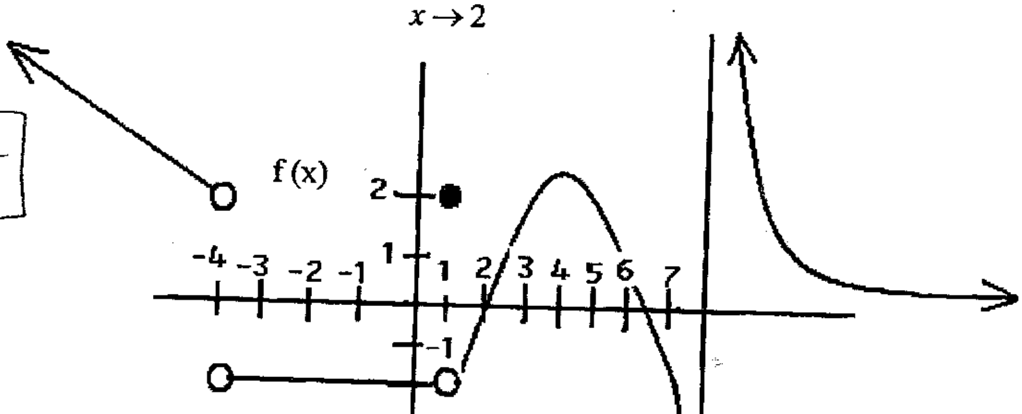
10) $\lim\limits_{x\to 5} \dfrac{1}{x-5}$ $\boxed{\text{DNE}}$

11) $\lim\limits_{x\to\infty} \dfrac{4}{x+3}$ $\boxed{0}$

12) $\lim\limits_{x\to-\infty} x+3$ $\boxed{\text{DNE}}$ or $-\infty$

13) $\lim\limits_{x\to 2} f(x) = 4$ $\lim\limits_{x\to 2} g(x) = 3$ $h(x) = \dfrac{2f(x)}{g(x)}$ *find* $\lim\limits_{x\to 2} h(x)$ $= \boxed{\frac{8}{3}}$

14) $\lim\limits_{x\to-\infty} \dfrac{x^2-4x}{4x^2-1} = \boxed{\frac{1}{4}}$



Given the sketch, find the requested limits.

15) $\lim\limits_{x\to\infty} f(x) = \boxed{0}$

16) $\lim\limits_{x\to 1} f(x) = \boxed{-2}$

17) $\lim\limits_{x\to -4} f(x) = \boxed{\text{DNE}}$

18) $\lim\limits_{x\to 8} f(x)$ $\boxed{\text{DNE}}$

19) $\lim\limits_{x\to -4^-} f(x)$ $\boxed{2}$

20) $\lim\limits_{x\to -4^+} f(x)$ $\boxed{-2}$

21) $f(-4)=$ $\boxed{\text{und}}$

Continuity

1) The five functions below are discontinuous at x = 1.

- For each function determine whether the discontinuity is removable or non-removable.
- Find $\lim_{x\to 1} f(x)$ if it exists.

(a) $f(x)=\dfrac{x^2+x-2}{x-1}$

Removable; $\lim_{x\to 1} f(x)=3$

(b) $f(x)=\begin{cases}\dfrac{x^2-1}{x-1}, & x\neq 1\\ 3, & x=1\end{cases}$

Removable; $\lim_{x\to 1} f(x)=2$

(c) $f(x)=\dfrac{|x-1|}{x-1}$

Non removable.

(d) $f(x)=\dfrac{1}{x-1}$

Non removable

(e) $f(x)=\cos\left(\dfrac{x}{x-1}\right)$

Nonrem.

2) Prove that f(x) is continous at x = 5. $f(x)=\begin{cases}\dfrac{x-5}{x^2-4x-5} & x\neq 5\\ \dfrac{1}{\sqrt{31+x}} & x=5\end{cases}$

$f(5)=\dfrac{1}{\sqrt{31+5}}=\dfrac{1}{6}$

$\lim_{x\to 5} f(x)=\lim_{x\to 5}\dfrac{(\cancel{x-5})}{(\cancel{x-5})(x+1)}=\lim_{x\to 5}\dfrac{1}{x+1}=\dfrac{1}{5+1}=\dfrac{1}{6}$

$\lim_{x\to 5} f(x)=f(5)=\dfrac{1}{6}$

$\therefore$ f(x) is con- @ x=5.

3) $f(x)=\begin{cases}3x-5 & x\neq 1\\ 2 & x=1\end{cases}$ Is this function continuous at x=1? Justify your answer.

$f(1)=2$, $\lim_{x\to 1} f(x)=\lim_{x\to 1} 3x-5=3(1)-5=-2$

The function f(x) is <u>not</u> continuous at x=1 because $f(1)\neq\lim_{x\to 1} f(x)$.

CONTINUITY EQUATION PROBLEMS

Find the value of the constant a that will make the function continuous where the defining rule changes.

1) f(x) = $\begin{cases} -0.4x-2, x \le 1 \\ 0.3x-a, x > 1 \end{cases}$

2) f(x) = $\begin{cases} 9-x^2, x < 2 \\ ax, x \ge 2 \end{cases}$

$9-4=2a$

$\boxed{a=5/2}$

3) f(x) = $\begin{cases} ax^2, x \le 3 \\ ax-3, x > 3 \end{cases}$

$9a=3a-3$

$6a=-3$

$\boxed{a=-\frac{1}{2}}$

4) f(x) = $\begin{cases} -0.4x+a^2, x < 1 \\ ax-2.4, x \ge 1 \end{cases}$

$-.4+a^2=a-2.4$

$a^2-a+2=0$

DNE

5) f(x) = $\begin{cases} x^2, x < 2 \\ a-x, x \ge 2 \end{cases}$

$4=a-2$

$\boxed{a=6}$

6) f(x) = $\begin{cases} 0.4x+1, x < 1 \\ ax+2, x \ge 1 \end{cases}$

$.4+1=a+2$

$1.4=a+2$

$\boxed{-.6=a}$

7) f(x) = $\begin{cases} ax-5, x < -1 \\ ax^2, x \ge -1 \end{cases}$

$-a-5=a$

$-5=2a$

$\boxed{a=-5/2}$

8) f(x) = $\begin{cases} a^2-x^2, x < 2 \\ 1.5ax, x \ge 2 \end{cases}$

$a^2-4=3a$

$a^2-3a-4=0$

$(a-4)(a+1)=0$

$\boxed{a=4 \text{ or } a=-1}$

Derivatives

C P M I
I M X P
P X

Derivatives using Graphs

1. Determine what is happening at the critical number 0 for the function:

$$f(x) = -\frac{x^4}{24} + \frac{x^2}{2} - 1 + \cos x. \quad f'(x) = -\frac{1}{6}x^3 + x - \sin x$$

f': $-$ | $+$
f: dec | 0 | inc

$\boxed{\text{min}}$

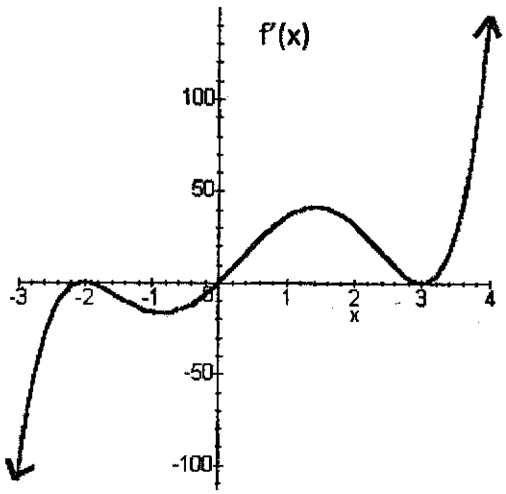
2. Given the graph of the derivative of f, answer the following questions.

a) Where is f increasing?

$f'(x) > 0$; $(0,3)(3,\infty)$

b) Where is f decreasing?

$f'(x) < 0$; $(-\infty,-2)(-2,0)$



c) Where does f have local maxima?

—

d) Where does f have local minima?

$x = 0$

e) Which is larger $f(2)$ or $f(3)$?

$f(3)$

f) Which is larger $f(-1)$ or $f(-3)$?

$f(-3)$

g) Is there an inflection point on f at $x = 3$?

yes

h) Is there an inflection point on f at $x = 0$?

NO

i) Is there an inflection point on f at $x = -2$?

yes

j) What is f'(3)?

0

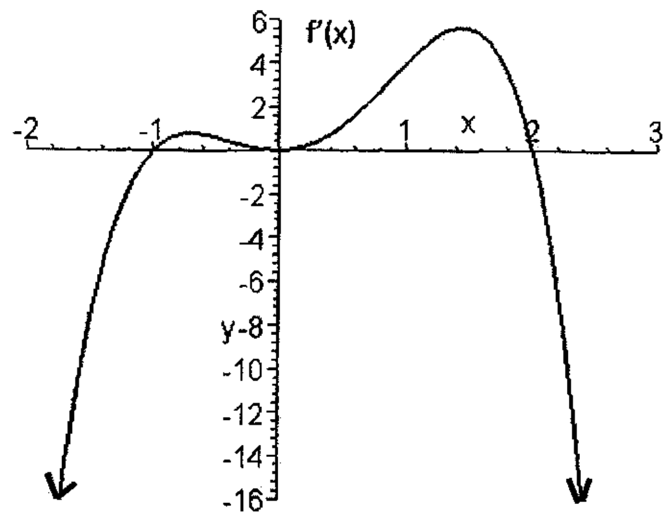
k) What is f'(o)?

0

3. Given the graph of the ***derivative*** of f, answer the following questions.

a) Where is f increasing?

$(-1,0)(0,2)$



b) Where is f decreasing?

$(-\infty,-1)(2,\infty)$

c) Where does f have local maxima?

$x = 2$

d) Where does f have local minima?

$x = -1$

e) Which is larger $f(0)$ or $f(1)$?

$f(1)$

f) Which is larger $f(-2)$ or $f(0)$?

$f(-2)$

g) Is there an inflection point at $x = 2$?

NO

h) Is there an inflection point at $x = 0$?

yes

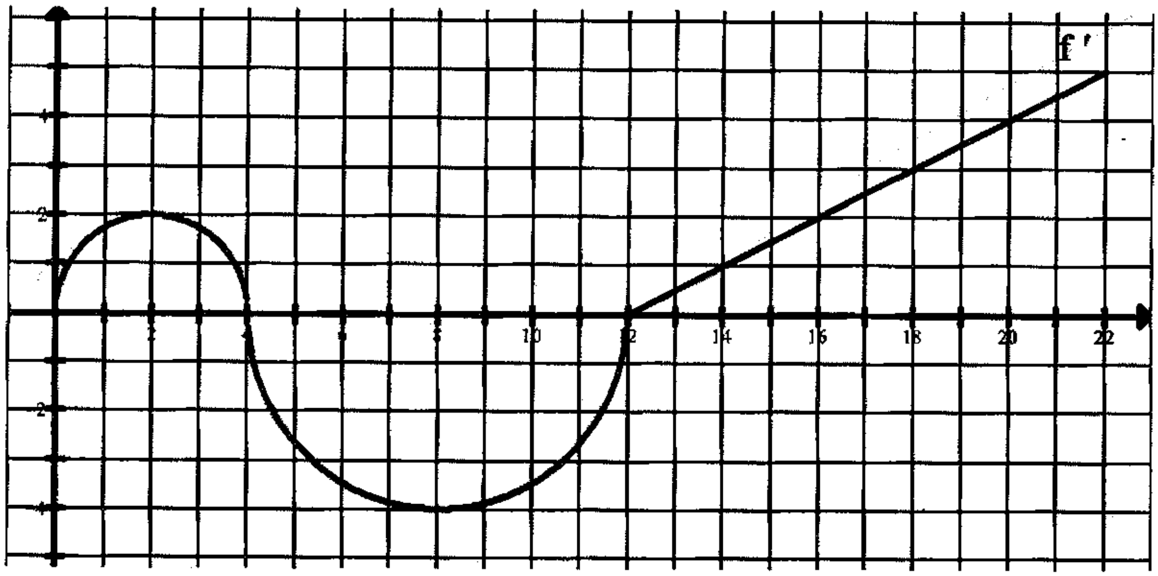
i) Is there an inflection point at $x = -1$?

NO

j) Write an equation for the line tangent to the graph of f at x=1 if f(1) = 3.

$y-3=4(x-1)$ $\quad$ $m=f'(1)=4$

The graph below shows f ', the derivative of function f. The graph consists of two semi-circles and one line segment. Horizontal tangents are located at x = 2 and x = 8 and a vertical tangent is located at x = 4.



a) On what intervals is f increasing? Justify your answer.

(0,4)(12,22) because $f'(x) > 0$ on these intervals

b) For what values of x does f have a relative minimum? Justify.

X=12 because

c) On what intervals is f concave up? Justify.

(0,2)(8,12)(12,22) because $f'(x)$ is inc

d) For what values of x is f " undefined?

$x=4$, $x=12$

e) Identify the x-coordinates for all points of inflection of f.

$x=2$, $x=8$

f) For what value of x does f reach its maximum value? Justify.

$x=4$ b/c

g) If f(4) = 5, find f(12).

$$f(12) = f(4) + \underbrace{\int_4^{12} f'(x)\,dx}_{\text{Area of semi-circle}} = 5 + \tfrac{1}{2}\pi(4)^2 = \boxed{5+8\pi}$$

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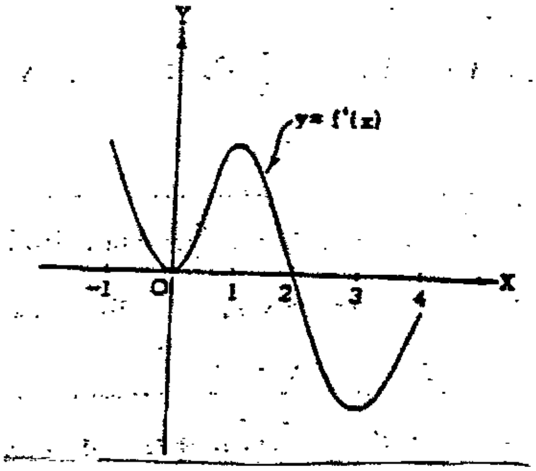
First and Second Derivative tests for extrema; Concavity

1.



Graph of f'

Let f be a function that has domain the closed interval [- 1, 4] and range the closed interval [- 1, 2]. Let $f(-1)=-1$, $f(0)=0$, and $f(4)=1$. Also let f has the derivative function f' that is continuous and that has the graph shown in the figure above.

(a) Find all values of x for which f assumes a relative maximum. Justify your answer.

$X=2$; $f'(x)>0$ on $[-1,2)$ and $f'(x)<0$ on $(2,4]$

(b) Find all values of x for which f assumes its absolute minimum. Justify your answer.

$X=-1$

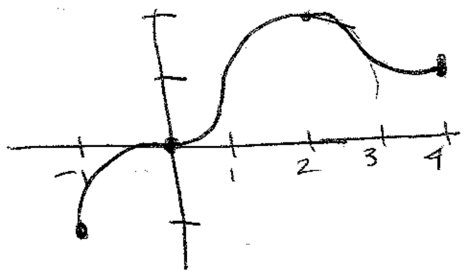
(c) Find the intervals on which f is concave downward.

$(-1,0)$ $(1,3)$

(d) Give all values of x for which f has a point of inflection.

$X=0$, $X=1$, $X=3$

(e) Sketch the graph of f.



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2. A function f is continuous on the closed interval [- 3, 3] such that $f(-3)=4$ and $f(3)=1$. The functions f' and f'' have the properties given in the table below.

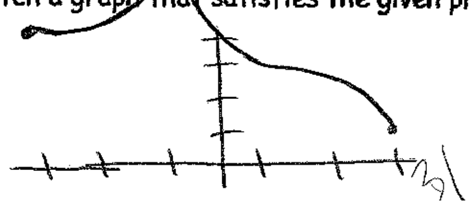
x	$-3<x<-1$	$x=-1$	$-1<x<1$	$x=1$	$1<x<3$
$f'(x)$	Positive	Fails to exist	Negative	0	Negative
$f''(x)$	Positive	Fails to exist	Positive	0	Negative

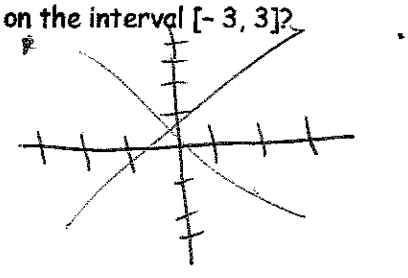
(a) What are the x-coordinates of all absolute maximum and absolute minimum points of f on the interval [- 3, 3]? Justify your answer.

Abs max $X=-1$; Abs min $X=3$

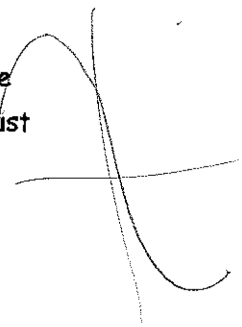
(b) What are the x-coordinates of all points of inflection of f on the interval [- 3, 3]? Justify your answer. $X=1$

(c) Sketch a graph that satisfies the given properties of f.





5. If f is a continuous function defined for all real numbers x and if the maximum value of $f(x)$ is 5 and the minimum value of $f(x)$ is -7, then which of the following must be true?



I. The maximum value of $f(|x|)$ is 5.

II. The maximum value of $|f(x)|$ is 7. ✓

III. The minimum value of $f(|x|)$ is 0.

(A) I only (B) II only (C) I and II only (D) II and III only (E) I, II, and III

---

6. An equation of the line tangent to $y=x^3+3x^2+2$ at its point of inflection is

$y'=3x^2+6x$ $y''=6x+6$

$x=-1$

POI

(A) $y=-6x-6$ (B) $y=-3x+1$ (C) $y=2x+10$

(D) $y=3x-1$ (E) $y=4x+1$

$y-4=-3(x+1)$

$f(-1)=4$ $m=f'(-1)=-3$ $y-4=-3x-3$

---

7. If the graph of $y=x^3+ax^2+bx-4$ has a point of inflection at $(1,-6)$, what is the value of b?

$y'=3x^2+2ax+b$

$y''=6x+2a$

$6(1)+2a=0$

$2a=-6$

$a=-3$

(A) -3 (B) 0 (C) 1

(D) 3 (E) It cannot be determined from the information given.

$-6=(1)^3-3(1)^2+b(1)-4$ $b=0$

---

8. For what value of x does the function $f(x)=(x-2)(x-3)^2$ have a relative maximum?

(A) -3 (B) $-\frac{7}{3}$ (C) $-\frac{5}{2}$ (D) $\frac{7}{3}$ (E) $\frac{5}{2}$

---

9. Let f be the function defined by $f(x)=\begin{cases} x^3 & \text{for } x\le 0 \\ x & \text{for } x>0 \end{cases}$. Which of the following statements about f is true?

(A) f is an odd function.

(B) f is discontinuous at $x=0$.

(C) f has a relative maximum.

(D) $f'(0)=0$

(E) $f'(x)>0$ for $x\neq 0$.

---

10. If $f(x)=\sin\left(\frac{x}{2}\right)$, then there exists a number c in the interval $\frac{\pi}{2}<x<\frac{3\pi}{2}$ that satisfies the

conclusion of the Mean Value Theorem. Which of the following could be c?

(A) $\frac{2\pi}{3}$ (B) $\frac{3\pi}{4}$ (C) $\frac{5\pi}{6}$ (D) π (E) $\frac{3\pi}{2}$

Finding Extrema and Points of Inflection

A) Find the critical numbers
B) State the intervals where the function is increasing and decreasing
C) Find the relative maximum and minimum
D) Find the points of inflection of f
E) State the intervals where the function is concave up and concave down

1) $f(x) = x^3 + 3x^2 + 1$

2) $f(x) = x^5 - 5x^4 + 100$ $f'(x) = 5x^4 - 20x^3$ $f''(x) = 20x^3 - 60x^2$

$5x^3(x-4)=0$
$x=0$ $x=4$

f′: + − +
f: inc 0 dec 4 inc
Inc: $(-\infty,0)(4,\infty)$
Dec: $(0,4)$

max @ $x=0$
min @ $x=4$

$20x^2(x-3)=0$
$\cancel{x=0}$ $x=3$

f″: − − +
f: ↓ 0 ↓ 3 ↑
Conc↓: $(-\infty,0)(0,3)$
Conc↑: $(3,\infty)$

3) $f(x) = \dfrac{x-1}{x+3}$ $f'(x) = \dfrac{4}{(x+3)^2}$ $f''(x) = \dfrac{-8}{(x+3)^3}$

$(x+3)^2=0$
$x=-3$

f′: + +
f: inc −3 inc
Inc: $(-\infty,-3)(-3,\infty)$
Dec: never

no max/min

$(x+3)^3=0$
$x=-3$

f″: − +
f: ↓ −3 ↑
Conc↓: $(-\infty,-3)$
Conc↑: $(-3,\infty)$

4) $f(t) = (t+1)^2(t-5)$ $f'(t) = (t+1)(3t-9)$ $f''(t) = 6t-6$

$(t+1)(3t-9)=0$
$t=-1$ $t=3$

f′: + − +
f: inc −1 dec 3 inc
Inc: $(-\infty,-1)(3,\infty)$
Dec: $(-1,3)$

max @ $x=-1$
min @ $x=3$

$6t-6=0$
$t=1$

f″: − +
f: ↓ 1 ↑
Conc↓ $(-\infty,1)$
Conc↑ $(1,\infty)$

6) $f(x) = x^{2/3}(2x-5)$ $f'(x) = 2x^{2/3} + 2x^{-1/3}(2x-5) = \dfrac{10x-10}{3x^{1/3}}$ $f''(x) = \frac{8}{3}x^{-1/3} - \frac{2}{9}x^{-4/3}(2x-5) = \dfrac{20x+10}{9x^{4/3}}$

$\dfrac{10x-10}{3x^{1/3}}=0$
zero: $10x-10=0 \therefore x=1$
und: $3x^{1/3}=0 \therefore x=0$

f′: + − +
f: inc 0 dec 1 inc
Inc: $(-\infty,0)(1,\infty)$
Dec: $(0,1)$

max @ $x=0$
min @ $x=1$

$\dfrac{20x+10}{9x^{4/3}}=0$
zero: $20x+10=0 \therefore x=-\frac{1}{2}$
und: $9x^{4/3}=0 \therefore \cancel{x=0}$

f″: − + +
f: ↓ −1/2 ↑ 0 ↑
Conc↓ $(-\infty,-1/2)$
Conc↑ $(-1/2,0)(0,\infty)$

7) $f(x) = \sqrt{x^2+1}$ $f'(x) = \dfrac{x}{(x^2+1)^{1/2}}$ $f''(x) = (x^2+1)^{-1/2} - x^2(x^2+1)^{-3/2} = \dfrac{1}{(x^2+1)^{3/2}}$

$\dfrac{x}{(x^2+1)^{1/2}}=0$
zero: $x=0$
und: $(x^2+1)^{1/2}=0$
$\cancel{x^2=-1}$

f′: − +
f: dec 0 inc
Inc: $(0,\infty)$
Dec: $(-\infty,0)$

min @ $x=0$

$\dfrac{1}{(x^2+1)^{3/2}}=0$
none

none

8) $f(x) = 2\cos x - x$ on $[0,2\pi]$ $f'(x) = -2\sin x - 1$ $f''(x) = -2\cos x$

$-2\sin x - 1 = 0$
$\sin x = -1/2$
$x = \frac{7\pi}{6}$ $x = \frac{11\pi}{6}$

− + −
0 dec $\frac{7\pi}{6}$ inc $\frac{11\pi}{6}$ dec 2π
Inc $(7\pi/6, 11\pi/6)$
Dec $(0, 7\pi/6)(\frac{11\pi}{6}, 2\pi)$

max @ $x=\frac{11\pi}{6}$
min @ $x=\frac{7\pi}{6}$

$-2\cos x = 0$
$\cos x = 0$
$x = \frac{\pi}{2}, \frac{3\pi}{2}$

− + −
0 ↓ $\frac{\pi}{2}$ ↑ $\frac{3\pi}{2}$ ↓ 2π
Conc↓ $(0,\pi/2)(\frac{3\pi}{2},2\pi)$
Conc↑ $(\pi/2, 3\pi/2)$

VT

Derivatives Using Data

Use your calculator, and give decimal answers correct to **three** decimal places.

1. A roast turkey is taken from an oven and placed on a counter to cool. The table shows the temperature of the turkey at various times over a three hour period.

t (minutes)	0	30	60	90	120	150	180
$T(t)$ (° F)	185	150	130	110	95	88	79

(a) Use data from the table to find $T(90)$. Using appropriate units, explain the meaning of your answer. $T(90) = 110°F$. The temperature of the turkey after 90 minutes 110°F.

(b) Use data from the table to find $T^{-1}(150)$. Using appropriate units, explain the meaning of your answer. —omit—

(c) For $0 < t < 180$, must there be a time t when the temperature of the turkey is 125° F? Justify your answer. Yes, by the IVT, since the data is continuous from $0 < t < 180$, it will take on every value between 185°F and 79°F

(d) Use data from the table to find an approximation for $T'(75)$. Show the computations that lead to your answer. Using appropriate units, explain the meaning of your answer.
$T'(75) = \frac{T(90) - T(60)}{90-60} = \frac{110-130}{30} = -2/3$ °F/min. The turkey is cooling at a rate of 2/3°F/min at $t = 75$.

2. A hot cup of coffee is taken into a classroom and set on a desk to cool. The table shows the temperature of the coffee at various times over a twelve minute period.

t (minutes)	0	2	5	6	8	10	12
$T(t)$ (° F)	113	103	95	94	92	91	90

(a) For $0 < t < 12$, must there be a time t when the temperature of the coffee is 99° F? Justify your answer. yes, since the function is continuous then, by the IVT, there must be a time t for $0 < t < 12$ where the temp would be b/w 113° & 90°

(b) Use data from the table to find an approximation for $T'(3)$. Show the computations that lead to your answer. Using appropriate units, explain the meaning of your answer.
$T'(3) = \frac{T(5) - T(2)}{5-2} = \frac{95-103}{3} = -8/3$ °F/min

(c) Use data from the table to find the average rate of change of $T(t)$ on the time period $5 \le t \le 12$ hours. $\frac{T(12) - T(5)}{12-5} = \frac{90-95}{7} = -5/7$ °F/min

(d) A model for the temperature is given by $y(t) = \frac{1}{3}\left(270 + 70e^{-0.3t}\right)$, where $y(t)$ is measured in degrees Fahrenheit and t is measured in minutes. Find $y'(3)$. $y'(3) = -2.846$ °F/min.

(e) Use the model given in part (d) to find the average rate of change of $y(t)$ on the time period $5 \le t \le 12$ hours.

$$\frac{y(12) - y(5)}{12-5} = -.653 \text{ °F/min}$$

4. (Modification of 2001 AB 2/ BC 2)

The temperature, in degrees Celsius (°C), of the water in a pond is a differentiable function W of time t. The table below shows the water temperature as recorded every 3 days over a 15-day period.

t (days)	0	3	6	9	12	15
$W(t)$ (°C)	20	31	28	24	22	20

data is cont, the function will take on every value bet 20 & 31 & also 24 & 22.

(a) Based on values in the table, what is the smallest number of instances at which the temperature of the pond could equal 23°C on the open interval $0 < t < 15$? Justify your answer.
There are 2 instances, on the interval (0,3) & (9,12). By the IVT, since the

(b) Use data from the table to find an approximation for $W'(7)$. Show the computations that lead to your answer. Using appropriate units, explain the meaning of your answer.
$W'(7) = \frac{W(9)-W(6)}{9-6} = \frac{24-28}{3} = -\frac{4}{3}$ °C/day. The temp of the water was changing at a rate of $-4/3$ °C/day on $t=7$.

(c) A student proposes the function P, given by $P(t) = 20 + 10te^{(-t/3)}$, as a model for the temperature of the water in the pond at time t, where t is measured in days and $P(t)$ is measured in degrees Celsius. Find $P'(7)$. Using appropriate units, explain the meaning of your answer. $P'(t) = -1.293$ °C/day

5. (Modification of 2004 Form B AB 3/ BC 3)

A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time t minutes, where v is a differentiable function of t. Selected values of $v(t)$ for $0 \le t \le 40$ are shown in the table below.

t (min)	0	5	10	15	20	25	30	35	40
$v(t)$ (mpm)	7.0	9.2	9.5	7.0	4.5	2.4	2.4	4.3	7.2

(a) Based on the values in the table, what is the smallest number of instances at which the velocity of the plane could be 8 miles per minute on the open interval $0 < t < 40$? Justify your answer. 2 instance. The data is cont. ∴ v(t) will take on every value between v(0) & v(5) as well as b/w v(10) & v(15)

(b) Use data from the table to find an approximation for $v'(18)$. Show the computations that lead to your answer. Using appropriate units, explain the meaning of your answer.

b) $v'(18) = \frac{v(20)-v(15)}{20-15} = \frac{4.5-7}{5} = -.5$ mpm²

(c) Use data from the table to find the average acceleration of the particle over the time interval $5 \le t \le 25$ minutes. $\frac{v(25)-v(5)}{25-5} = \frac{2.4-9.2}{20} = -.34$ mpm²

(d) The function f, defined by $f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$, is used to model the velocity of the plane, in miles per minute, for $0 \le t \le 40$. According to this model, what is the acceleration of the plane at $t = 23$? Indicate units of measure. .0091 mpm²

(e) Use the model given in part (d) to find the average acceleration of the particle over the time interval $5 \le t \le 25$ minutes. .009

"Derivatives in Disguise"

Each of the following is ***f ' (a)*** for some function ***f*** and some number ***a***. Identify ***f (x)*** and ***a***. Evaluate each limit.

1. $\lim\limits_{h\to 0} \dfrac{(2+h)^{(3)}-8}{h} = \boxed{12}$

$f(x)=x^3 \quad f'(2)=3x^2\big|_{x=2}$

2. $\lim\limits_{h\to 0} \dfrac{\sin(3+h)-\sin(3)}{h} = \boxed{\cos(3)}$

$f(x)=\sin x \qquad f'(a)$

$a=3$

3. $\lim\limits_{h\to 0} \dfrac{\sqrt{(4+h)}-2}{h} = \boxed{\frac{1}{4}}$

$f(x)=\sqrt{x} \qquad \cancel{\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}} \qquad a=4$

$f'(4)=\frac{1}{2}x^{-1/2}\big|_{x=4}=\frac{1}{2}\left(\frac{1}{2}\right)$

4. $\lim\limits_{h\to 0} \dfrac{\tan(3\pi/4+h)+1}{h} = \boxed{2}$

$f(x)=\tan x \qquad a=\frac{3\pi}{4}$

$f'\left(\frac{3\pi}{4}\right)=\sec^2 x\big|_{x=3\pi/4}=\left[\sec\frac{3\pi}{4}\right]^2$

5. $\lim\limits_{h\to 0} \dfrac{(2)^{(4+h)}-16}{h} = \boxed{16\ln 2}$

$f(x)=2^x \quad a=4$

$f'(4)=2^x\ln 2\big|_{x=4}$

6. $\frac{1}{5}\lim\limits_{h\to 0} \dfrac{\ln(e+h)-1}{\cancel{5}h} = \boxed{\frac{1}{5e}}$

hint: $\lim\limits_{x\to a}\dfrac{f(x)}{k}=\dfrac{1}{k}\bullet\lim\limits_{x\to a}f(x)$

$f(x)=\frac{1}{5}\ln x \quad a=e \quad f'(e)=\frac{1}{5x}\big|_{x=e}=\frac{1}{5}\cdot\frac{1}{e}$

7. $\lim\limits_{x\to 9} \dfrac{\sqrt{x}-3}{x-9} = \boxed{\frac{1}{6}}$

$f(x)=\sqrt{x}$

hint: $\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$

$a=9 \qquad f'(9)=\frac{1}{2}x^{-1/2}\big|_{x=9}$

8. $\lim\limits_{x\to \pi/2} \dfrac{3\sin(x)-3}{x-\pi/2} = \boxed{0}$

hint: $\lim\limits_{x\to a}k\bullet f(x)=k\bullet\lim\limits_{x\to a}f(x)$

$f(x)=3\sin x \quad a=\pi/2$

$f'(\pi/2)=3\cos x\big|_{x=\pi/2}=$

9. $\lim\limits_{x\to 0} \dfrac{5\cos(x)-5}{x} = \boxed{0}$

$f(x)=5\cos x \quad a=0$

$f'(0)=-5\sin x\big|_{x=0}=$

More Derivatives in Disguise

The given limit is a derivative

a) Of what function ?
b) At what x value?
c) Evaluate

1) $\lim\limits_{h\to 0}\dfrac{.25(3+h)^4-.25(3)^4}{h}$

a) $f(x)=.25x^4$
b.) $x=3$
c) $x^3\Big|_{x=3}=\boxed{27}$

2) $\lim\limits_{h\to 0}\dfrac{3(1+h)-3}{h}$

a.) $f(x)=3x$
b.) $x=1$
c.) $\boxed{3}$

3) $\lim\limits_{h\to 0}\dfrac{\sqrt{(1+h)^3}-1}{h}$

a) $f(x)=x^{3/2}$
b.) $x=1$
c.) $\frac{3}{2}x^{\frac{1}{2}}\Big|_{x=1}=\boxed{\frac{3}{2}}$

4) $\lim\limits_{h\to 0}\dfrac{\tan(\frac{\pi}{4}+h)-1}{h}$

a.) $f(x)=\tan x$
b.) $x=\pi/4$
c.) $\sec^2 x\Big|_{x=\pi/4}=\left(\frac{2}{\sqrt{2}}\right)^2=\boxed{2}$

5) $\lim\limits_{h\to 0}\dfrac{\frac{1}{8+h}-\frac{1}{8}}{h}$

a.) $f(x)=\frac{1}{x}$
b) $\cancel{x}\, x=8$
c.) $-x^{-2}\Big|_{x=8}=\boxed{\frac{-1}{64}}$

6) $\lim\limits_{h\to 0}\dfrac{\cos(\frac{3\pi}{2}+h)}{h}$

a.) $f(x)=\cos x$
b.) $x=3\pi/2$
c.) $-\sin x\Big|_{x=\frac{3\pi}{2}}=\boxed{+1}$

7) $\lim\limits_{h\to 0}\left(\frac{1}{\sqrt{5+h}}-\frac{1}{\sqrt{5}}\right)\left(\frac{1}{h}\right)$

a.) $f(x)=\frac{1}{\sqrt{x}}$
b.) $x=5$
c.) $-\frac{1}{2}x^{-3/2}\Big|_{x=5}=\boxed{-\frac{1}{2\sqrt{125}}}$

8) $\lim\limits_{x\to 2}\dfrac{2x^3-16}{x-2}$

a.) $f(x)=2x^3$
b.) $x=2$
c.) $6x^2\Big|_{x=2}=\boxed{24}$

Piecewise Function Differentiability

I. Is the function differentiable at the point where the rule for the function changes?

1. $f(x) = \begin{cases} x^2-6x+8, x \ge 1 \\ 7-4x, x<1 \end{cases}$ LHD: -4 RHD: $2x-6\Big|_{x=1} = -4$

$1-6+8=3$ ✓
$7-4=3$
$\boxed{\text{YES}}$

2. $f(x) = \begin{cases} 3-2x, x<2 \\ 3x-7, x \ge 2 \end{cases}$ LHD: -2 RHD: 3 $\therefore \boxed{\text{NO}}$

3. $f(x) = \begin{cases} 5-6x, x \le 3 \\ -4-x^2, x>3 \end{cases}$ LHD: -6 RHD: $-2x\Big|_{x=3} = -6$

$5-6(3) = -13$
$-4-(3)^2 = -13$
$\boxed{\text{YES}}$

4. $f(x) = \begin{cases} x^2-4x+8, x \le 3 \\ 11-x, x>3 \end{cases}$ LHD: $2x-4\Big|_{x=3} = 2$ RHD: -1

$\therefore \boxed{\text{NO}}$

5. $f(x) = \begin{cases} x^2-x+1, x \le 1 \\ 2-x, x>1 \end{cases}$ LHD: $2x-1\Big|_{x=1} = 1$ RHD: -1 $\boxed{\text{NO}}$

6. $f(x) = \begin{cases} \frac{1}{2}x+1, x<2 \\ \sqrt{2x}, x \ge 2 \end{cases}$ LHD: $\frac{1}{2}$ RHD: $\frac{1}{2}(2x)^{-\frac{1}{2}} \cdot 2\Big|_{x=2} = \frac{1}{2}$

$\frac{1}{2}(2)+1=2$
$\sqrt{2 \cdot 2} = 2$
$\boxed{\text{YES}}$

7. $f(x) = \begin{cases} x^2-4, x<2 \\ \sqrt{x-2}, x \ge 2 \end{cases}$

LHD: $2x\Big|_{x=2} = 4$
RHD: $\frac{1}{2}(x-2)^{-1/2}\Big|_{x=2} = \text{und}$
$\boxed{\text{NO}}$

II. Find a and k that make the functions differentiable at the point where the rule for the function changes.

8. $f(x) = \begin{cases} \frac{a}{x}, x \le 1 \\ 12-kx^2, x>1 \end{cases}$

$-\frac{a}{x^2} = -2kx$
$-a = -2k$
$a = 2k$

$2k = 12-k(1)$
$\cancel{3k=12}$
$\boxed{k=4}$ $\boxed{a=8}$

9. $f(x) = \begin{cases} ax^2+10, x<2 \\ x^2-6x+k, x \ge 2 \end{cases}$

$2ax = 2x-6$
$4a = -2$
$\boxed{a = -\frac{1}{2}}$

$-\frac{1}{2}(2)^2+10 = 4-12+k$
$8 = -8+k$
$\boxed{k=16}$

10. $f(x) = \begin{cases} x^3, x<1 \\ a(x-2)^2+k, x \ge 1 \end{cases}$

$3x^2 = 2a(x-2)$
$3 = -2a$
$\boxed{a = -\frac{3}{2}}$

$(1)^3 = -\frac{3}{2}(1-2)^2+k$
$1 = -\frac{3}{2}+k$
$\boxed{k = \frac{5}{2}}$

11. $f(x) = \begin{cases} k^2-x^2, x<2 \\ -akx+5, x \ge 5 \end{cases}$

skip

Tell whether the function is continuous at the value of x where the rule for the function changes. Tell whether the derivative exists at that value.

12. $f(x) = \begin{cases} 7-x^2, x \le 2 \\ 5-x, x>2 \end{cases}$

$f(2) = 7-2^2 = 3$

$\lim_{x \to 2^-} f(x) = 7-2^2 = 3$
$\lim_{x \to 2^+} f(x) = 5-2 = 3$
$\Big\} \lim_{x \to 2} f(x) = 3$

LHD: $-2x\Big|_{x=2} = -4$
RHD: -1
$\therefore$ f(x) is continuous at x=2 but not differentiable

Using the Rules

1) Find y' if $y = \sin 2x$

$y' = \cos 2x \cdot 2 = \boxed{2\cos 2x}$

2) Find y' if $y = \cos^2(x+2) = [\cos(x+2)]^2$

$y' = 2\cos(x+2) \cdot -\sin(x+2) \cdot 1$

$y' = \boxed{-2\cos(x+2)\sin(x+2)}$

3) Find y' if $y = 29x - 5$

$\boxed{y' = 29}$

4) Find y' if $y = \arctan(2x)$

$y' = \frac{1}{1+(2x)^2} \cdot 2 = \boxed{\frac{2}{1+4x^2}}$

5) Find y' if $y = \frac{2x}{3}$

$\boxed{y' = 2/3}$

6) Find y' if $y = \sqrt{x^5} = x^{5/2}$

$\boxed{y' = \frac{5}{2}x^{3/2}}$

7) Find y' if y = xsecx

$y' = x \cdot \sec x \tan x + \sec x(1)$

$\boxed{y' = x\sec x \tan x + \sec x}$

8) Find y' if $y = \frac{x^2+1}{2x-1}$

$y' = \frac{(2x-1)(2x) - (x^2+1)(2)}{(2x-1)^2} = \boxed{\frac{2x^2-2x-2}{(2x-1)^2}}$

9) Find y' if $y = (3x^2+3)^2$

$y' = 2(3x^2+3)(6x) = \boxed{12x(3x^2+3)}$

10) Find y' if $y = \frac{1}{\sqrt{4x+1}}$

$y' = -\frac{1}{2}(4x+1)^{-3/2}(4) = \boxed{\frac{-2}{(4x+1)^{3/2}}}$

11) Find y' if $y = \ln(3x^3+8)$

$y' = \frac{1}{3x^3+8} \cdot 9x^2 = \boxed{\frac{9x^2}{3x^3+8}}$

12) Find y' if $y = 4^{x^2-2}$

$4^{x^2-2} \ln(4)(2x)$

$\boxed{4^{x^2-2}[2x\ln(4)]}$

13. $y = (x^2+1)^5$

$y' = 5(x^2+1)^4(2x) = \boxed{10x(x^2+1)^4}$

14. $y = 5x^3 - 3x^5$

$\boxed{y' = 15x^2 - 15x^4}$

15. $y = (3x-1)(2x+5)$

$y' = (3x-1)(2) + (2x+5)(3)$

$y' = 6x - 2 + 6x + 15 = \boxed{12x+13}$

16. $y = (2x+3)^2$

$y' = 2(2x+3)(2) = \boxed{4(2x+3)}$

17. $y = \sqrt{2x+1}$

$y' = \frac{1}{2}(2x+1)^{-1/2} \cdot 2 = \boxed{\frac{1}{\sqrt{2x+1}}}$

18. $y = (x + \frac{1}{x})^2$

$y' = 2(x+\frac{1}{x})(1+(-x^{-2})) = \boxed{2(x+\frac{1}{x})(1-\frac{1}{x^2})}$

19. $y = x^2(x^3-1) = x^5 - x^2$

$\boxed{y' = 5x^4 - 2x}$

20. $y = \ln(3x+4)$

$y' = \frac{1}{3x+4} \cdot 3 = \boxed{\frac{3}{3x+4}}$

21) $y = \ln(3x^5)$

$y' = \frac{1}{3x^5} \cdot 15x^4 = \boxed{\frac{5}{x}}$

22) $y = \ln(e^{3x})$

$y' = \frac{1}{e^{3x}} \cdot e^{3x} \cdot 3 = \boxed{3}$

23) $y = \ln(\sin 4x)$

$y' = \frac{1}{\sin 4x} \cdot \cos 4x \cdot 4 = \boxed{4\cot 4x}$

24) $y = \ln(e^5)$

$\boxed{y' = 0}$

25) $y = e^{x^3}$ $y' = e^{x^3} \cdot 3x^2 = \boxed{3x^2 e^{x^3}}$

26) $y = e^{5\ln x}$

$y' = e^{5\ln x} \cdot \frac{5}{x} = \boxed{\frac{5}{x}e^{5\ln x}}$

27) $y = e^{\cos x}$

$y' = e^{\cos x} \cdot -\sin x = \boxed{-\sin x\, e^{\cos x}}$

Implicit Differentiation

Find $\frac{dy}{dx}$ by implicit differentiation:

1. $y = x^3$

$\boxed{\frac{dy}{dx} = 3x^2}$

2. $x = y^3$

$1 = 3y^2 \frac{dy}{dx}$ $\boxed{\frac{dy}{dx} = \frac{1}{3y^2}}$

3. $y = x + 3y$

$\frac{dy}{dx} = 1 + 3\frac{dy}{dx}$

$-2\frac{dy}{dx} = 1$

$\boxed{\frac{dy}{dx} = -\frac{1}{2}}$

4. $y = xy^2$

$\frac{dy}{dx} = x \cdot 2y\frac{dy}{dx} + y^2(1)$

$\frac{dy}{dx} - 2xy\frac{dy}{dx} = y^2$

$\boxed{\frac{dy}{dx} = \frac{y^2}{1-2xy}}$

5. $y^3 + x^3 - y^2 - 5y - x^2 = -4$

$3y^2\frac{dy}{dx} + 3x^2 - 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$

$\frac{dy}{dx}(3y^2 - 2y - 5) = 2x - 3x^2$

$\boxed{\frac{dy}{dx} = \frac{2x-3x^2}{3y^2-2y-5}}$

6. $2x^6y^9 + 3x - y^3 = 52^3$

$12x^5y^9 + 9y^8\frac{dy}{dx} \cdot 2x^6 + 3 - 3y^2\frac{dy}{dx} = 0$

$\frac{dy}{dx}(18x^6y^8 - 3y^2) = 3 - 12x^5y^9$

$\boxed{\frac{dy}{dx} = \frac{3-12x^5y^9}{18x^6y^8-3y^2}} = \boxed{\frac{1-4x^5y^9}{6x^6y^8-y^2}}$

7. $\cos(2y) = x$

$-\sin(2y) \cdot 2\frac{dy}{dx} = 1$ $\frac{dy}{dx} = \frac{1}{-2\sin(2y)} = \boxed{-\frac{1}{2}\csc(2y)}$

8. Find the slope of the tangent line to $x^2 + 4y^2 = 4$ at the point $(\sqrt{2}, \frac{-1}{\sqrt{2}})$ and write the equation of the tangent line.

$2x + 8y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{8y}\Big|_{(-\sqrt{2}, -1/\sqrt{2})} = \frac{-2\sqrt{2}}{8(-\frac{1}{\sqrt{2}})} = \frac{-\sqrt{2}}{4} \cdot -\sqrt{2} = \boxed{\frac{1}{2}}$

$\boxed{y + \frac{1}{\sqrt{2}} = \frac{1}{2}(x - \sqrt{2})}$

9. Find $\frac{d^2y}{dx^2}$ for $1 - xy = x - y$

$0 - [x\frac{dy}{dx} + y(1)] = 1 - \frac{dy}{dx}$

$-x\frac{dy}{dx} - y = 1 - \frac{dy}{dx}$

$\frac{dy}{dx}(1-x) = 1 + y$

$\frac{dy}{dx} = \frac{1+y}{1-x}$

$\frac{d^2y}{dx^2} = \frac{(1-x)(\frac{dy}{dx}) - (1+y)(-1)}{(1-x)^2}$

$= \frac{\cancel{(1-x)}(\frac{1+y}{\cancel{1-x}}) + 1 + y}{(1-x)^2}$

$\boxed{\frac{d^2y}{dx^2} = \frac{2+2y}{(1+x)^2}}$

10. Find the equation of the tangent line and the normal line. $x^2 + y^2 = 9$ for $(0,3)$, $(2, \sqrt{5})$

$2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x}{y}\Big|_{(0,3)} = -\frac{0}{3} = 0$

$y - 3 = 0(x - 0)$

$\boxed{y = 3}$

$\frac{dy}{dx} = \frac{-x}{y}\Big|_{(2,\sqrt{5})} = \frac{-2}{\sqrt{5}}$

$\boxed{y - 3 = \frac{-\sqrt{5}}{2}(x - 2)}$

Derivative Rules Using Data

Directions: Apply the derivative rule first. Evaluate using given information. Give your final answer.

Given: $g(3) = 5$ $g'(3) = 4$ $g'(-2) = 3$

$h(3) = -2$ $h'(3) = 7$ $h'(5) = 6$

Find $f'(3)$ for each of the following.

1) $f(x) = \dfrac{g(x)}{h(x)}$

2) $f(x) = h(x) - g(x)$

$f'(x) = 7 - 4 = \boxed{3}$

3) $f(x) = h(g(x))$

4) $f(x) = [h(x)]^4$

$f'(x) = 4(h(x))^3 \cdot h'(x)$

$= 4(-2)^3 \cdot 7 = \boxed{-224}$

5) $f(x) = g(x)\ h(x)$

$f'(3) = 5(7) + (-2)(4)$

$= \boxed{27}$

6) $f(x) = g(x) + h(x)$

$f'(3) = 4 + 7 = \boxed{11}$

7) $f(x) = \dfrac{h(x)}{g(x)}$

$f'(3) = \dfrac{5(7) - (-2)(4)}{5^2} = \boxed{\dfrac{43}{25}}$

8) $f(x) = g(h(x))$

$f'(3) = g'(h(3)) \cdot h'(3)$

$= 3 \cdot 7$

$= \boxed{21}$

9) $f(x) = [g(x)]^3$

$f'(3) = 3(g(x))^2 g'(x)$

$= 3(25) \cdot 4$

$= \boxed{300}$

Use the table to find $h'(x)$ for the following:

x	f(x)	f'(x)	g(x)	g'(x)
-3	2	4	6	11
0	1	2	-3	-7
1	16	-3	4	13
2	-3	2	0	$\sqrt{2}$

10) $h(x) = f(g(x))$ at $x = 0$

$h'(0) = f'(g(0)) \cdot g'(0)$

$= 4 \cdot -7 = \boxed{-28}$

11) $h(x) = f(x)g(x)$ at $x = 1$

$h'(1) = f(1)g'(1) + g(1)f'(1)$

$= 16(13) + 4(-3) = \boxed{196}$

12) $h(x) = f(x) + g(x)$ at $x = -3$

$h'(3) = 4 + 11 = \boxed{15}$

13) $h(x) = 3f(x) - g(x)$ at $x = 2$

14) $h(x) = [f(x)]^2$ at $x = 1$

$h'(1) = 2f(1) \cdot f'(1)$

$= 2(16) \cdot (-3)$

$= \boxed{-96}$

15) $h(x) = g(f(x))$ at $x = -3$

$h'(-3) = g'(f(3)) \cdot f'(3)$

$= \sqrt{2} \cdot 4$

$= \boxed{4\sqrt{2}}$

Related Rates

$\frac{2}{4} = \frac{r_1}{h_1}$ $2h = 4r$ $r = \frac{h}{2}$ $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$ $\boxed{V = \frac{1}{12}\pi h^3}$

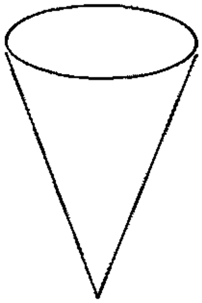
RELATED RATE EXAMPLES AND NOTES

$r = 2$ $h = 4$ $\frac{dV}{dt} = 200\pi \text{ in}^3/\text{min}$

Want $\frac{dh}{dt}$ when $h = 3$

<ins>Volume of a Cone</ins>

$V = \frac{1}{3}\pi r^2 h$



$\frac{dV}{dt} = \frac{1}{12}\pi \cdot 3h^2 \frac{dh}{dt}$

$200\pi = \frac{\pi}{4}(3)^2 \frac{dh}{dt}$

$200 = \frac{9}{4}\frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = 88.889 \text{ in/min}}$

Things to remember:

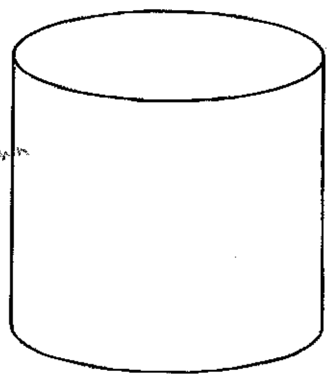
1) Diameter = 2 x radius
2) As the volume increases or decreases both the radius and the height change. The radius and the height are always proportional to each other.
$\frac{r}{h} = \frac{r_1}{h_1}$
3) Radius and height are measured in linear units. Volume is measured in cubic units.
4) If the volume is decreasing, the rate of change is negative.

Water is flowing into a conical cup with a radius of 2 inches and a height of 4 inches. How fast is the height of the water in the cup increasing when the height is 3 inches? The volume of the cup is increasing at a rate of 200π in^3 per minute.

Volume of a Cylinder

Cyl: $r = 10$, $h = 22$ $h = 8$ $\frac{dh}{dt} = 2 \text{ in/min}$

$V = \pi r^2 h$ Want $\frac{dV}{dt}$



Things to remember:

1) Radius = ½ diameter

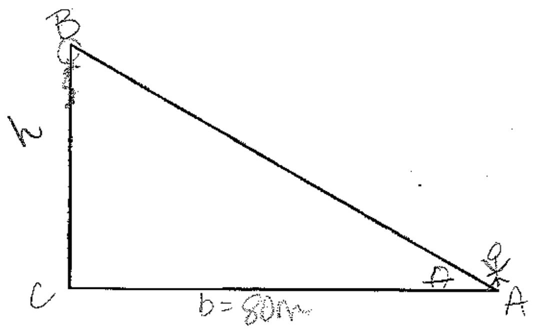
2) Radius of a cylinder is constant.

3) Radius and height are measured in linear units. Volume is measured in cubic units.

4) If the volume or height is decreasing, the rate of change is negative.

Mrs. AP Calculusteacher has a headache from her too loud calculus class. She goes to the math office to take some aspirin. The water cooler is cylindrical in shape with a radius of 10 inches and a height of 22 inches. She notices that the height of the water in the tank is 8 inches and is dropping at a rate of 2 inches per minute. How fast is the water flowing out of the tank?

<u>**Rate of Change of θ**</u>



Things to remember:

1) $\text{Tan } \theta = \dfrac{\text{opposite}}{\text{adjacent}}$

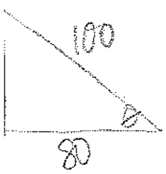
2) The derivative of tangent θ is $\sec^2 \theta$.

3) Height is measured in linear units. θ is measured in radians.

Dorothy, who is standing at point A, watches the Professor in the balloon rise straight up from point C. Dorothy is 80 meters east of point C. The balloon is rising at a constant rate of 5 meters per second. Find the rate of change of θ when the balloon is 60 meters off the ground.

$\frac{dh}{dt} = 5 \text{ m/sec}$

Find $\frac{d\theta}{dt}$ when $h = 60$



$\sec\theta = \frac{100}{80}$

$\tan\theta = \frac{h}{80}$

$h = 80\tan\theta$

$\frac{dh}{dt} = 80\sec^2\theta \frac{d\theta}{dt}$

$5 = 80\left(\frac{100}{80}\right)^2 \frac{d\theta}{dt}$

$5 = 125 \frac{d\theta}{dt}$

$\boxed{\frac{d\theta}{dt} = \frac{1}{25} \text{ rad/sec}}$

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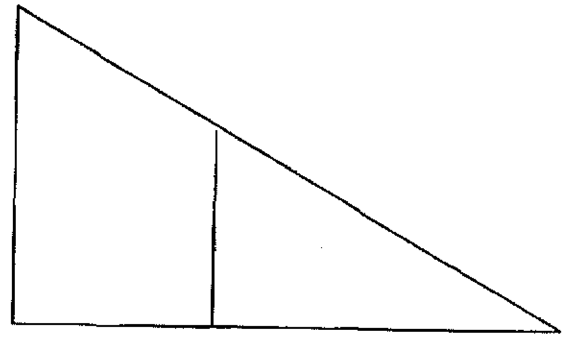
Shadow Problems: Similar Triangles



Things to remember:

1) Corresponding parts of similar triangles are proportional.

2) $\frac{h_1}{b_1} = \frac{h_2}{b_2}$

3) The tip of the shadow is the entire base of the large triangle.

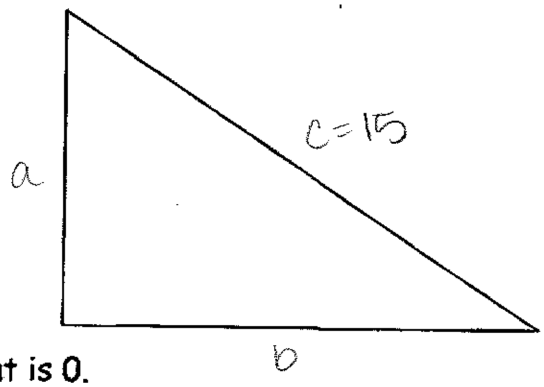
4) The man's shadow is the base of the small triangle.

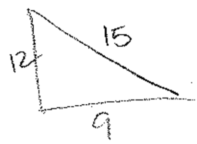
A man 6 feet tall walks at a rate of 5 feet per second away from a light that is 15 feet above the ground. When he is 10 feet from the base of the light, a) at what rate is the tip of his shadow changing? b) at what rate is the length of his shadow changing?

Pythagorean Problem

Formula:

$a^2 + b^2 = c^2$





Things to remember:

1) The derivative of a constant is 0.
2) If distance decreases, rate is negative.

Barney tricked the cat into climbing to the top of a 15-foot ladder leaning against the side of a building. Barney began pulling the ladder away from the building at a rate of ½ foot per second. When the ladder is 9 feet from the base of the building, how fast is the cat plummeting toward the ground?

$C=15$

$\frac{db}{dt} = 1/2$ ft/sec

Find $\frac{da}{dt}$ when $b=9$

$a^2+b^2=15$

$2a\frac{da}{dt} + 2b\frac{db}{dt} = 0$

$2(12)\frac{da}{dt} + 2(9)\left(\frac{1}{2}\right) = 0$

$24\frac{da}{dt} = -9$

44 $\boxed{\frac{da}{dt} = \frac{-3}{8} \text{ ft/sec}}$

DERIVATIVE OF THE INVERSE OF A FUNCTION

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$.

$$g'(x) = \frac{1}{f'(g(x))}$$

Part 1 Given a function f(x). g(x) is the inverse of the function. Find g'(x).

1. $f(x) = 2x + \cos x$ $g(1) = 0$ Find $g'(1)$
2. $f(x) = x^3 + x + 1$ $g(1) = 0$ Find $g'(1)$
3. $f(x) = x^5 - x^3 + 2x$ $g(2) = 1$ Find $g'(2)$
4. $f(x) = x^3$ $g(8) = 2$ Find $g'(8)$
5. $f(x) = \sqrt{x-2}$ $g(2) = 4$ Find $g'(2)$
6. $f(x) = \dfrac{1}{x-1}$ $g(2) = \dfrac{3}{2}$ Find $g'(2)$

Part 2 g(x) is the inverse of f(x). Find g(a) and then find g'(a)

7) $f(x) = \sqrt{3x+1}$ $a = 1$

8) $f(x) = x^2 - 16,\ x \geq 0$ $a = 9$

9) $f(x) = \sqrt{4-x}$ $a = 3$

10) $f(x) = x^3 + 5$ $a = -3$

11) $f(x) = 3x^5 + 2x^3$ $a = 5$

12) $f(x) = \sin x,\ -\pi/2 < x < \pi/2$ $a = ½$

13) $f(x) = 2x^2 + 8x + 7,\ x \leq -2$ $a = 1$

Integrals

Riemann Sums/Trapezoidal Sums

1. A table of values for f(t) is given.

t	0	20	40	60	80	100	120
f(t)	1.2	2.8	4.0	4.7	5.1	5.2	4.8

a) Estimate $\int_0^{120} f(t)dt$ by using a left Riemann sun with six subintervals.

$$\int_0^{120} f(t)\,dt \approx 20(1.2+2.8+4+4.7+5.1+5.2) = \boxed{460}$$

b) Estimate $\int_0^{120} f(t)dt$ by using a right Riemann sun with six subintervals.

$$\int_0^{120} f(t)\,dt \approx 20(2.8+4+4.7+5.1+5.2+4.8) = \boxed{532}$$

c) Estimate $\int_0^{120} f(t)dt$ by using a midpoint sum with three subintervals.

$$\int_0^{120} f(t)\,dt \approx 40(2.8+4.7+5.2) = \boxed{508}$$

d) Estimate $\int_0^{120} f(t)dt$ by using the trapezoidal rule with three subintervals.

$$\int_0^{20} f(t)\,dt \approx \frac{40}{2}\left[1.2+2(4+5.1)+4.8\right] = \boxed{484}$$

2. A table of values for g(t) is given.

t	0	40	70	90	100
g(t)	150	180	195	184	172

a) Estimate $\int_0^{100} g(t)dt$ by using a left Riemann sum with four subintervals.

$$\int_0^{100} g(t)\,dt \approx 40(150)+30(180)+20(195)+10(184) = \boxed{17{,}140}$$

b) Estimate $\int_0^{100} g(t)dt$ by using a right Riemann sum with four subintervals

$$\int_0^{100} g(t)\,dt \approx 40(180)+30(195)+20(184)+10(172) = \boxed{18{,}450}$$

c) Estimate the average value of g(t) from 0 to 100 using a trapezoidal sum with four subintervals

$$\text{avg}(t) = \frac{1}{100-0}\int_0^{100} g(t)\,dt = \frac{1}{100}\left[\frac{40}{2}(150+180)+\frac{30}{2}(180+195)+\frac{20}{2}(195+184)+\frac{10}{2}(184+172)\right]$$

$$= \boxed{177.95}$$

d) Estimate g'(55) using values in the table

$$g'(55) = \frac{g(70)-g(40)}{70-40} = \frac{195-180}{30} = \boxed{\frac{1}{2}}$$

Evaluating Definite and Indefinite Integrals

1. $\int 3dx$

$\boxed{3x + C}$

2. $\int s^6 ds$

$\boxed{\frac{s^7}{7} + C}$

3. $\int x^{1/5} dx$

$\boxed{\frac{5}{6}x^{6/5} + C}$

4. $\int 3u^7 du$

$\boxed{\frac{3u^8}{8} + C}$

5. $\int \frac{-6}{x^2}dx = \int -6x^{-2}\,dx = 6x^{-1} + C$

$= \boxed{\frac{6}{x} + C}$

6. $\int \frac{10}{x}dx = 10\int \frac{1}{x}\,dx$

$= \boxed{10\ln x + C}$

7. $\int \sqrt[3]{x}dx = \int x^{\frac{1}{3}}dx = \boxed{\frac{3}{4}x^{\frac{4}{3}} + C}$

8. $\int (x + \frac{1}{x})dx = \boxed{\frac{x^2}{2} + \ln x + C}$

9. $(4t^3 - 3t^{1/4} + 3)dt = \frac{4t^4}{4} - 3\cdot\frac{4}{5}t^{5/4} + 3t + C$

$= \boxed{t^4 - \frac{12}{5}t^{5/4} + 3t + C}$

10. $\int (5x^4 - 3x^{5/2} + \frac{2}{5x^4})dx$

$= \frac{5x^5}{5} - 3x^{7/2}\cdot\frac{2}{7} + \frac{2}{5}\cdot\frac{x^{-3}}{-3} + C = \boxed{x^5 - \frac{6}{7}x^{\frac{7}{2}} - \frac{2}{15x^3} + C}$

11. $\int \frac{x^3 + 2x^2}{3x^5}dx = \int \frac{1}{3x^2} + \frac{2}{3x^3}\,dx$

$= \frac{1}{3}\cdot\frac{x^{-1}}{-1} + \frac{2}{3}\cdot\frac{x^{-2}}{-2} + C$

$= \boxed{-\frac{1}{3x} - \frac{1}{3x^2} + C}$

12. $\int \sqrt{x}(x+3)dx = \int x^{3/2} + 3x^{\frac{1}{2}}\,dx$

$= \boxed{\frac{2}{5}x^{5/2} + 2x^{3/2} + C}$

13. $\int_{-1}^{2} dx = x\Big]_{-1}^{2} = 2 - (-1) = \boxed{3}$

14. $\int_{1}^{2} (x^4 - 10)dx = \frac{x^5}{5} - 10x\Big]_{1}^{2}$

$= (\frac{32}{5} - 20) - (\frac{1}{5} - 10) = \boxed{-\frac{19}{5}}$

15. $\int_{1}^{3} \frac{1}{x^2}dx = -x^{-1}\Big]_{1}^{3} = -\frac{1}{3} - (-1) = \boxed{\frac{2}{3}}$

16. $\int_{1}^{4} \sqrt{x}dx = \frac{2}{3}x^{\frac{3}{2}}\Big]_{1}^{4} = \frac{2}{3}(8) - \frac{2}{3} = \boxed{\frac{14}{3}}$

17. $\int_{-1}^{2} (4 - x^5)dx = 4x - \frac{x^6}{6}\Big]_{-1}^{2}$

$= (8 - \frac{64}{6}) - (-4 - \frac{1}{6}) = \boxed{\frac{3}{2}}$

18. $\int_{4}^{9} \frac{1}{\sqrt{t}}dt = 2t^{\frac{1}{2}}\Big]_{4}^{9} = 2(3) - 2(2) = \boxed{2}$

19. $\int_{\pi}^{3\pi/2} \cos x dx = \sin x\Big]_{\pi}^{3\pi/2} = -1 - 0 = \boxed{-1}$

20. $\int 2\sec^2 x dx = \boxed{2\tan x + C}$

21. $\int \frac{1}{2}\sin x dx = \boxed{-\frac{1}{2}\cos x + C}$

22. $\int_{0}^{\pi/2} 2\cos x dx = 2\sin x\Big]_{0}^{\pi/2}$

$= 2(1) - 2(0) = \boxed{2}$

Using Substitution to Evaluate Integrals

1) $\int(1+2x)^4(2)dx$ $u=1+2x$, $du=2dx$

$=\int u^4\,du$

$=u^5/5+C$

$=\boxed{\dfrac{(1+2x)^5}{5}+C}$

2) $\int(x^2-9)^3(2x)dx$ $u=x^2-9$, $du=2x\,dx$

$\int u^3\,du$

$=\dfrac{u^4}{4}+C$

$=\boxed{\dfrac{(x^2-9)^4}{4}+C}$

3) $\int\sqrt{9-x^2}(-2x)dx$ $u=9-x^2$, $du=-2x\,dx$

$\int u^{1/2}\,du$

$=\boxed{\frac{2}{3}(9-x^2)^{3/2}+C}$

4) $\int x^3(x^4+3)^2dx$ $u=x^4+3$, $du=4x^3dx$

$=\frac{1}{4}\int u^2\,du$

$=\frac{1}{4}\cdot\frac{u^3}{3}+C$

$=\boxed{\dfrac{(x^4+3)^3}{12}+C}$

5) $\int x^2(x^3-1)^4dx$ $u=x^3-1$, $du=3x^2dx$

$\frac{1}{3}\int u^4du$

$=\boxed{\dfrac{(x^3-1)^5}{15}+C}$

6) $\int x\sqrt{x^2+2}dx$ $u=x^2+2$, $du=2x\,dx$

$=\frac{1}{2}\int u^{1/2}du$

$=\boxed{\dfrac{(x^2+2)^{3/2}}{3}+C}$

7) $\int 5x\sqrt[3]{1-x^2}dx$ $u=1-x^2$, $du=-2x\,dx$

$5\cdot-\frac{1}{2}\int u^{3/2}du$

$-\frac{5}{2}\cdot\frac{2}{5}u^{5/2}+C=\boxed{-(1-x^2)^{5/2}+C}$

8) $\int x\sqrt{x^2+2}dx$

same as #6

9) $\int\dfrac{x}{(1-x^2)^3}dx$ $u=1-x^2$, $du=-2x\,dx$

$-\frac{1}{2}\int\frac{1}{u^3}\,du$

$-\frac{1}{2}\cdot\frac{u^{-2}}{-2}+C=\boxed{\dfrac{1}{4(1-x^2)^2}+C}$

10) $\int\dfrac{x^2}{(1+x^3)^2}dx$ $u=1+x^3$, $du=3x^2dx$

$\frac{1}{3}\int u^{-2}\,du$

$\frac{1}{3}\cdot\frac{u^{-1}}{-1}+C=\boxed{-\dfrac{1}{3(1+x^3)}+C}$

11) $\int\dfrac{x}{\sqrt{1-x^2}^3}dx$ $u=1-x^2$, $du=-2x\,dx$

$-\frac{1}{2}\int u^{-3/2}\,du$

$=-\frac{1}{2}\cdot-2u^{-1/2}+C=\boxed{\dfrac{1}{\sqrt{1-x^2}}+C}$

12) $\int(1+\frac{1}{t})^3(\frac{1}{t^2})dt$ $u=1+\frac{1}{t}$, $du=-\frac{1}{t^2}dt$

$-\int u^3du$

$=\boxed{\dfrac{-(1+\frac{1}{t})^4}{4}+C}$

13) $\int\sec2x\tan2x\,dx$ $u=2x$, $du=2dx$

$\frac{1}{2}\int\sec u\tan u\,du$

$\boxed{\frac{1}{2}\sec 2x+C}$

14) $\int x\cos x^2dx$ $u=x^2$, $du=2x\,dx$

$\frac{1}{2}\int\cos u\,du$

$\boxed{\frac{1}{2}\sin x^2+C}$

15) $\int\sin5x\,dx$ $u=5x$, $du=5dx$

$\frac{1}{5}\int\sin u\,du$

$\boxed{-\frac{1}{5}\cos5x+C}$

16) $\int\cos\frac{x}{2}dx$ $u=\frac{x}{2}$, $du=\frac{1}{2}dx$

$2\int\cos u\,du$

$\boxed{2\sin(\frac{x}{2})+C}$

17) $\int\frac{1}{3}\sec^2 8x\,dx$ $u=8x$, $d=8dx$

$\frac{1}{3}\cdot\frac{1}{8}\int\sec^2u\,du$

$\boxed{\frac{1}{24}\tan(8x)+C}$

18) $\int\cos^3x\sin x\,dx$ $u=\cos x$, $du=\sin x\,dx$

$\int u^3du$

$\boxed{\dfrac{\cos^4x}{4}+C}$

19) $\int\sqrt{\cos6x}\sin6x\,dx$ $u=\cos6x$, $du=-6\sin6x\,dx$

$-\frac{1}{6}\int u^{1/2}du$

$-\frac{1}{6}\cdot\frac{2}{3}u^{3/2}+C$

$\boxed{-\frac{1}{9}(\cos6x)^{3/2}+C}$

20) $\int\dfrac{\sin x}{(4-\cos x)^3}dx$ $u=4-\cos x$, $du=\sin x\,dx$

$\int u^{-3}du$

$\boxed{-\dfrac{1}{2(4-\cos x)^2}+C}$

Fundamental Theorem of Calculus

Find F'(x)

1. $F(x)=\int_{-2}^{x}(t^2-2t+5)dt$ $\quad F'(x)=x^2-2x+5$

2. $F(x)=\int_{1}^{x}\sqrt[4]{t}\,dt$ $\quad F'(x)=\sqrt[4]{x}$

3. $F(x)=\int_{x}^{x+2}(4t+1)dt$ $\quad =\int_0^{x+2}(4t+1)dt-\int_0^{x}(4t+1)dt$ $\quad F'(x)=4(x+2)+1-(4x+1)$

4. $F(x)=\int_{0}^{\sin x}\sqrt{t}\,dt$ $\quad F'(x)=\sqrt{\sin x}\cdot\cos x$

5. $F(x)=\int_{2}^{x^2}\frac{1}{t^2}dt$ $\quad F'(x)=\frac{1}{x^4}\cdot 2x=\frac{2}{x^3}$

6. $F(x)=\int_{0}^{x^3}\sin t^2\,dt$ $\quad F'(x)=\sin(x^6)\cdot 3x^2$

#3 cont: $4x+8+1-4x-1=8$

Use the Second Fundamental Theorem of Calculus to solve:

7. $y'=2+\frac{1}{x^2}$ and $y(1)=6$. Find $y(3)$. $\quad y(3)=6+\int_1^3(2+x^{-2})dx=6+\left[2x-\frac{1}{x}\right]_1^3$

$=6+\left(\frac{17}{3}-1\right)=6+\frac{14}{3}=\boxed{\frac{32}{3}}$

8. $f'(x)=\cos(x)$ and $f(0)=3$. Find $f\left(\frac{\pi}{4}\right)$.

$f(\pi/4)=3+\int_0^{\pi/4}\cos x\,dx=3+\sin x\Big]_0^{\pi/4}=3+\left(\frac{\sqrt{2}}{2}-0\right)=\boxed{3+\frac{\sqrt{2}}{2}}$

9. Water flows into a tank at a rate of $\frac{dW}{dt}=\frac{1}{75}\left(600+20t-t^2\right)$, where $\frac{dW}{dt}$ is measured in gallons per hour and t is measured in hours. If there are 150 gallons of water in the tank at time $t=0$, how many gallons of water are in the tank when $t=24$?

$150+\frac{1}{75}\int_0^{24}600+20t-t^2\,dt=150+\frac{1}{75}\left(600t+10t^2-t^3/3\right)\Big|_0^{24}=\boxed{357.36\text{ gallons}}$

Work problems 10-11 using the Fundamental Theorem of Calculus and your calculator.

10. $f'(x)=\cos\left(x^3\right)$ and $f(0)=2$. Find $f(1)$. $\quad f(1)=2+\int_0^1\cos(x^3)dx=\boxed{2.932}$

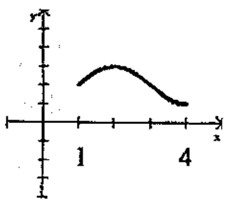
11. $f'(x)=e^{-x^2}$ and $f(5)=1$. Find $f(2)$. $\quad f(2)=1+\int_5^2 e^{-x^2}dx=1-\int_2^5 e^{-x^2}dx$

$=\boxed{.996}$

Use the Fundamental Theorem of Calculus and the given graph.

12. The graph of f' is shown on the right.

$\int_1^4 f'(x)dx=6.2$ and $f(1)=3$. Find $f(4)$.



$f(4)=3+\int_1^4 f'(x)dx$

$=3+6.2=9.2$

13. If $f(1)=12$, f' is continuous, and $\int_1^4 f'(x)\,dx=17$, what is the value of $f(4)$?

$f(4)=12+\int_1^4 f'(x)\,dx=12+17=29$

14. If $\int_2^5 (2f(x)+3)\,dx=17$, find $\int_2^5 f(x)\,dx$.

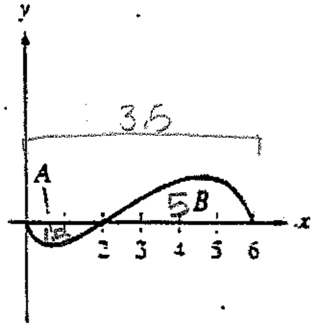
$2\int_2^5 f(x)\,dx+\int_2^5 3\,dx=17$

$2\int_2^5 f(x)\,dx+9=17$

$2\int_2^5 f(x)\,dx=8$

$\int_2^5 f(x)\,dx=4$

15. Region A has an area of 1.5, and $\int_0^6 f(x)\,dx=3.5$. Find:



(a) $\int_2^6 f(x)\,dx=\int_0^6 f(x)\,dx-\int_0^2 f(x)\,dx$

$=3.5-(-1.5)=5$

(b) $\int_0^6 |f(x)|\,dx=1.5+5=6.5$

16. Given the values of the derivative $f'(x)$ in the table and that $f(0)=100$, estimate $f(x)$ for $x=2, 4, 6$. Use a right Riemann sum.

$f(2)=100+\int_0^2 f'(x)\,dx$

$=100+[2(18)]=136$

x	0	2	4	6
$f'(x)$	10	18	23	25

$f(4)=100+\int_0^4 f'(x)\,dx$

$=100+[2(18)+2(23)]=182$

$f(6)=100+\int_0^6 f'(x)\,dx=100+[2(18+23+25)]=232$

17. Consider the function f that is continuous on the interval $[-5,5]$ and for which $\int_0^5 f(x)\,dx=4$. Evaluate:

(a) $\int_0^5 (f(x)+2)\,dx=\int_0^5 f(x)\,dx+\int_0^5 2\,dx$

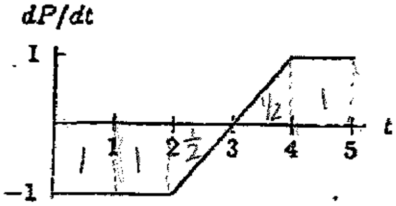
$=4+10=14$

(b) $\int_{-2}^3 \underbrace{f(x+2)}_{\text{shift left 2}}\,dx=4$

(c) $\int_{-5}^5 f(x)\,dx$ (f is even) $=2\int_0^5 f(x)\,dx=2(4)=8$

(d) $\int_{-5}^5 f(x)\,dx$ (f is odd) $=0$

18. Use the figure on the right and the fact that $P(0)=2$ to find values of P when $t=1, 2, 3, 4$, and 5.

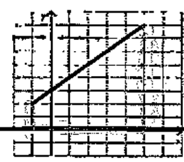


$P(5)=2+(-1)=1$

$P(1)=2+(-1)=1$ $P(3)=2+(-2.5)=-.5$

$P(2)=2+(-2)=0$ $P(4)=2+(-2)=0$

19. The graph of g' is shown on the right. Find $g(5)$ given that $g(-1)=-8$.



$g(5)=-8+\underbrace{\int_{-1}^5 g'(x)\,dx}_{\text{area of trapezoid}}=-8+\frac{6}{2}(2+8)$

$=22$

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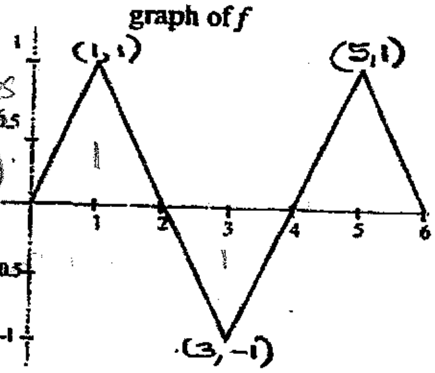
FUNCTIONS DEFINED BY INTEGRALS

$g'(x) = f(x)$

1. The function g is defined on the interval [0, 6] by $g(x) = \int_0^x f(t)\,dt$ where f is the function graphed in the figure.



(a) For what values of $x, 0 < x < 6$, does g have a relative maximum? Justify your answer. at $x=2$; $g'(x)=f(x)$ changes from pos. to neg.

(b) For what values of x is the graph of g concave down? Justify your answer. $(1,3)(5,6)$ where $f(x)$ is dec

(c) Write an equation for the tangent line to g at the point where $x = 3$. $g(3)=\int_0^3 f(t)dt = 1-\frac{1}{2}=\frac{1}{2}$ $g'(3)=f(3)=-1$

(d) Sketch a graph of the function g. List the coordinates of all critical point and inflection points.

$\therefore \boxed{y-\frac{1}{2}=-1(x-3)}$

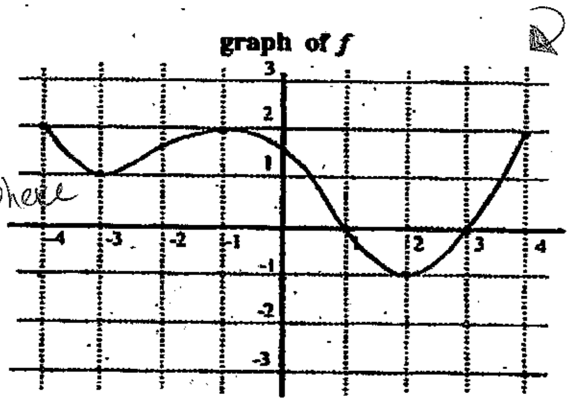
---

2. Suppose that f' is a continuous function, that $f(1) = 13$, and that $f(10) = 7$. Find the average value of f' over the interval [1, 10]. $\frac{1}{9}\int_1^{10} f'(x)dx = \frac{1}{9}[f(10)-f(1)] = \frac{1}{9}(7-13) = \boxed{-\frac{2}{3}}$

---

3. The graph of a differentiable function f on the closed interval $[-4, 4]$ is shown. $G'(x) = f(x)$

Let $G(x) = \int_{-4}^{x} f(t)\,dt$ for $-4 \le x \le 4$.



(a) Find $G(-4)$. $= \int_{-4}^{-4} f(t)\,dt = \boxed{0}$

(b) Find $G'(-4)$. $= f(-4) = \boxed{2}$

(c) On which interval or intervals is the graph of G decreasing? Justify your answer. $(1,3)$, where $f(x) < 0$

(d) On which interval or intervals is the graph of G concave down? Justify your answer. $(-4,-3)$ $(-1,2)$

(e) For what values of x does G have an inflection point? Justify your answer. $x = -3, -1, 2$

---

4. The function F is defined for all x by $F(x) = \int_0^{x^2} \sqrt{t^2+8}\,dt$.

(a) Find $F'(x)$. $= \sqrt{x^4+8} \cdot 2x = \boxed{2x\sqrt{x^4+8}}$

(b) Find $F'(1)$. $= 2(1)\sqrt{1+8} = 2(3) = \boxed{6}$

(c) Find $F''(x)$. $2x \cdot \frac{1}{2}(x^4+8)^{-1/2}(4x^3) + \sqrt{x^4+8} \cdot 2 = \boxed{\frac{4x^4}{\sqrt{x^4+8}} + 2\sqrt{x^4+8}}$

(d) Find $F''(1)$. $\frac{4}{3} + 2(3) = \frac{4}{3} + 6 = \boxed{\frac{22}{3}}$

---

5. If $F(x) = \int_x^{-5} (t^2 - t - 6)\,dt$, on what intervals is F decreasing?

$F(x) = -\int_{-5}^{x} (t^2-t-6)\,dt$

$F'(x) = -(x^2-x-6) = -x^2+x+6$

$-x^2+x+6=0$

$-(x^2-x-6)=0$

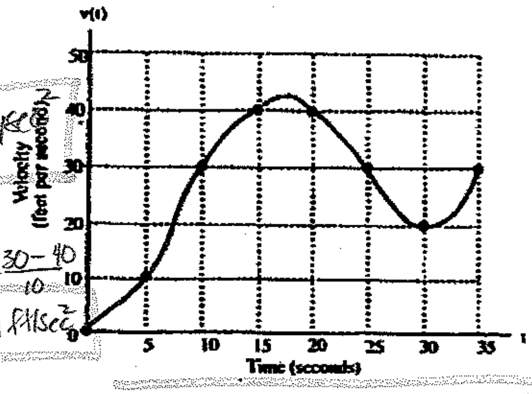
$-(x-3)(x+2)=0$

F': $-$ | $+$ | $-$

F: -2, 3

$\boxed{(-\infty,-2)\ (3,\infty)}$

6. The graph of the velocity $v(t)$, in ft/sec, of a car traveling on a straight road, for $0 \le t \le 35$, is shown in the figure.



(a) Find the average acceleration of the car, in ft/sec^2, over the interval $0 \le t \le 35$. $\frac{V(35)-V(0)}{35} = \frac{30-0}{35} = \boxed{\frac{6}{7} \text{ ft/sec}^2}$

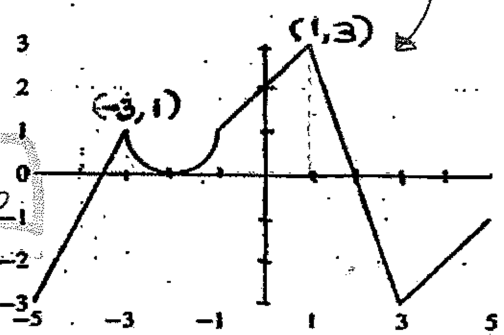
(b) Find an approximation for the acceleration of the car, in ft/sec^2, at $t = 20$. Show your computations. $a(20) = v'(20) \approx \frac{V(25)-V(15)}{10} = \frac{30-40}{10} = \boxed{-1 \text{ ft/sec}^2}$

(c) Approximate $\int_5^{35} v(t)\,dt$ with a Riemann sum, using the midpoints of three subintervals of equal length. Explain the meaning of this integral.

$\int_5^{35} v(t)\,dt = 10(30+40+20) = 10(90) = 900$ The car traveled 900 ft from t=0 to t=35 sec.

7. The function F is defined for all x by $F(x) = \int_0^x f(t)\,dt$, where f is the function graphed in the figure. The graph of f is made up of straight lines and a semicircle.

$F'(x) = f(x)$



(a) For what values of x is F decreasing? Justify your answer. $\boxed{(-5,-3)\,(2,5)}$

(b) For what values of x does F have a local maximum? A local minimum? Justify your answer. max $x=2$ min $x=-3.5$

(c) Evaluate $F(2)$, $F'(2)$, and $F''(2)$.

(d) Write an equation of the line tangent to the graph of F at $x = 4$.

(e) For what values of x does F have an inflection point? Justify your answer.

c) $F(2) = \int_0^2 f(t)\,dt = \frac{1}{2}(2+3) + \frac{1}{2}(1)(3) = \frac{5}{2} + \frac{3}{2} = \boxed{4}$

$F'(2) = f(2) = \boxed{0}$

$F''(2) = f'(2) = \boxed{-3}$ (slope of the line @ $x=2$)

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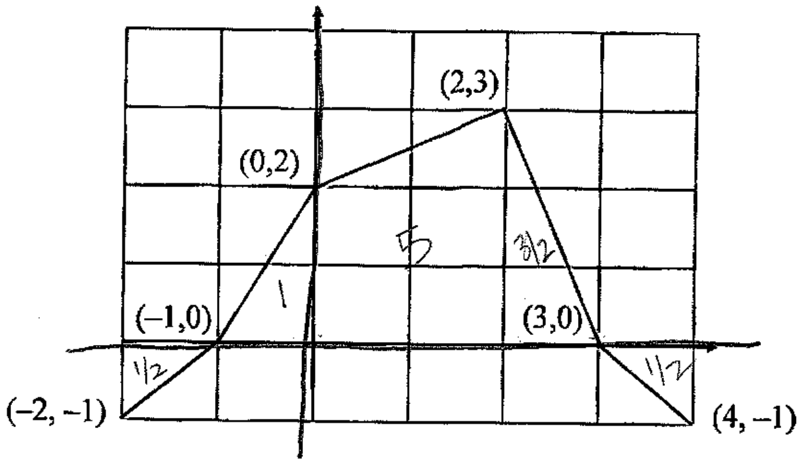
The Detective's Hat Function



The graph of the function f shown above is a piecewise continuous function defined on $[-2, 4]$. The graph of f consists of five line segments.

Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

1. Find each of the following. a) $g(-2) = \int_0^{-2} f(t)dt = -\int_{-2}^{0} f(t)dt = -(1/2) = -\frac{1}{2}$

   (a) $g(-2)$ (b) $g(-1)$ (c) $g(0)$ (d) $g(2)$ (e) $g(3)$ (f) $g(4)$

   $-1/2$ -1 0 5 $13/2$ 6

2. Explain the procedure you followed to answer question 1.

3. Find each of the following. $g'(x) = f(x)$

   (a) $g'(-1)$ (b) $g'(0)$ (c) $g'(2)$ (d) $g'(4)$

   0 2 3 -1

4. Explain the procedure you followed to answer question 3.

5. Explain why g must be a continuous function on $[-2, 4]$.

6. Write the equation for $g'(x)$ on the interval $[0, 2]$.

7. Write the equation for the line tangent to g at $x = 1$. Justify your answer.

8. Does $g''(0)$ exist? Explain your reasoning.

9. Will a point of inflection for g exist when $x = 0$? Explain your reasoning.

10. For what values of x in the open interval $(-2, 4)$ is g increasing? Explain your reasoning.

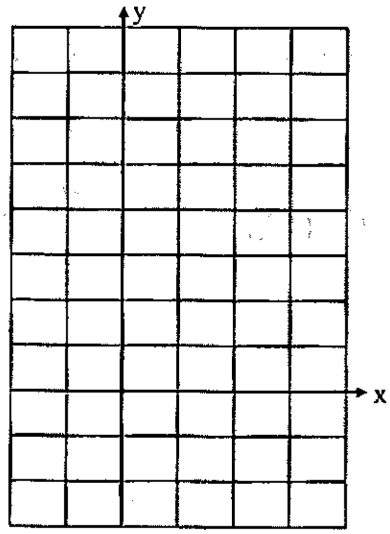
11. For what values of x in the open interval $(-2, 4)$ is g decreasing?

12. For what values of x in the open interval $(-2, 4)$ is g concave up? Explain your reasoning.

13. For what values of x in the open interval $(-2, 4)$ is g concave down?

14. Find the maximum and the minimum values of g on the closed interval $[-2, 4]$. Justify your answers.

15. On the axes provided, sketch the graph of function g on the closed interval $[-2, 4]$.



For questions 15 – 17, let h be the function given by $h(x)=\int_{-2}^{x} f(t)\,dt$.

16. Find each of the following.

(a) $h(-2)$ (b) $h(-1)$ (c) $h(0)$ (d) $h(2)$ (e) $h(3)$ (f) $h(4)$

17. Find the following and explain your reasoning.

(a) $h'(-1)$ (b) $h'(0)$ (c) $h'(2)$ (d) $h'(4)$

18. $g(x)-h(x)=k$, where k is a constant. Find the value of k and explain your reasoning.

19. If $w(x)=\int_{3}^{x} f(t)\,dt$, find $w(0)$.

20. $w(x)$ can also be defined as $w(x)=r+\int_{0}^{x} f(t)\,dt$ where r is a constant. What is the value of r?

Differential Equations

1) $\frac{dy}{dx}=\frac{1}{x}$ Find the general solution to this differential equation. Find the particular equation given that the point (1,2) is on the curve.

$\int dy=\int \frac{1}{x}dx$ $\quad 2=\ln 1+C$ $\quad \boxed{y=\ln x+2}$

$y=\ln x+C$ $\quad C=2$

2) $\frac{dy}{dx}=5x^4$ Find the general solution to this differential equation. Find the particular equation given that the point (0,2) is on the curve.

$\int dy=\int 5x^4 dx$

$y=x^5+C$ $\quad \boxed{y=x^5+2}$

$2=C$

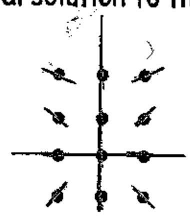
3) $\frac{dy}{dx}=\frac{1}{3y^2}$ Find the general solution to this differential equation. Find the particular equation given that the point (2, - 2) is on the curve.

$\int 3y^2 dy=\int dx$ $\quad y^3=x-10$

$y^3=x+C$ $\quad \boxed{y=\sqrt[3]{x-10}}$

$-8=2+C$ $\quad C=-10$

4) $\frac{dy}{dx}=\frac{2x}{y}$ Find the general solution to this differential equation and graph the slopefield below:



$\int y\,dy=\int 2x\,dx$ $\quad \boxed{y=\pm\sqrt{2x^2+C}}$

$\frac{y^2}{2}=x^2+C$

5) $\frac{dy}{dx}=\frac{3x^2}{e^{2y}}$ Find the general solution to this differential equation. Find the particular equation given that y(0) = 0.5

$\int e^{2y}dy=\int 3x^2 dx$ $\quad \frac{1}{2}e^{2y}=x^3+\frac{e}{2}$ $\quad 2y=\ln(2x^3+e)$

$\frac{1}{2}e^{2y}=x^3+C$ $\quad e^{2y}=2x^3+e$ $\quad \boxed{y=\frac{\ln(2x^3+e)}{2}}$

$1/2e=C$ $\quad \ln e^{2y}=\ln 2x^3+e$

6) $\frac{dy}{dx}=\frac{1+x}{xy}$ Find the particular solution given y(1) = -4

$\int y\,dy=\int \frac{1}{x}+1\,dx$ $\quad 8=\ln 1+1+C$ $\quad \frac{y^2}{2}=\ln x+x+7$

$y^2/2=\ln x+x+C$ $\quad 7=C$

$y=\pm\sqrt{2\ln x+2x+14}$

$\boxed{y=-\sqrt{2\ln x+2x+14}}$

7) $\frac{dy}{dx}=\frac{e^x}{y}$ Find the particular solution given y(0) = 4

$\int y\,dy=\int e^x dx$ $\quad \frac{y^2}{2}=e^x+C$ $\quad \frac{y^2}{2}=e^x+7$

$8=1+C$ $\quad C=7$ $\quad y=\pm\sqrt{2e^x+14}$ $\quad \boxed{y=\sqrt{2e^x+14}}$

8) $\frac{dy}{dx}=\frac{-4x+2}{y}$ Let y = f(x) be the particular solution to the differential equation with the initial condition f(2) = -4.

a) Write the equation for the line tangent to the graph at (2,-4) and use it to approximate f(1.8).

b) Find the particular solution y = f(x) to the differential equation with the initial condition f(2) = -4.

$m=\frac{-4(2)+2}{-4}=\frac{-3}{2}$ $\quad y+4=-3/2(x-2)$

$y+4=-3/2(1.8-2)$

$\boxed{f(1.8)\approx -3/2(1.8-2)-4}$

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b.) $\int y\,dy=\int -4x+2\,dx$ $\quad \frac{y^2}{2}=-2x^2+2x+12$

$\frac{y^2}{2}=-2x^2+2x+C$

$y=\pm\sqrt{-4x^2+4x+24}$

$8=-4+C$

$C=12$ $\quad \boxed{y=-2\sqrt{-x^2+x+6}}$

9) Population y grows according to the equation $\frac{dy}{dt} = ky$, where k is a constant and t is measured in years. If the population doubles every 10 years, then the value of k is

10) Let y(t) be the temperature, in degrees Fahrenheit of a cup of tea at time t minutes. Room temperature is 70 and the initial temperature of the tea is 180. The tea's temperature at time t is described by the differential equation $\frac{dy}{dt} = -0.1(y-70)$, with the initial condition y(0) = 180.

a) Use separation of variables to find an expression for y in terms of t, where t is measured in minutes.

$\int \frac{1}{y-70}\,dy = \int -.1\,dt$ $\quad \ln 110 = C$ $\quad y-70 = 110e^{-.1t}$

$\ln(y-70) = -.1t + C$ $\quad \ln(y-70) = -.1t + \ln 110$ $\quad \boxed{y = 110e^{-.1t} + 70}$

$e^{\ln(y-70)} = e^{-.1t} \cdot e^{\ln 110}$

b) How hot is the tea after 10 minutes?

$y(10) = 110e^{-.1(10)} + 70 \approx \boxed{110.467°\text{ F}}$

11) Let P(t) represent the number of wolves in a population at time t. The population P(t) is increasing at a rate directly proportional to 1000 - P, where the constant of proportionality is k.

a) If P(0) = 400, find P in terms of t and k.

b) If P(2) = 700, find k.

c) $\lim_{t\to\infty} P(t)$

12) The thickness y(t) of ice forming on a lake satisfies the differential equation $y'(t) = \frac{3}{y(t)}$ where t is measured in hours.

$y(t)\,dy = 3\,dt$

a) If y(0) = 1, find y(t)

b) When is the thickness two inches?

13) Oil is being pumped continuously from a certain oil well at a rate proportional to the amount of oil left in the well: that is, $\frac{dy}{dx} = ky$, where y is the amount of oil left in the well at any time t. Initially there were 1,000,000 gallons of oil in the well and 6 years later there were 500,000 gallons remaining. It will no longer be profitable to pump oil when there are fewer than 50,000 gallons remaining.

a) Write an equation for y, the amount of oil remaining in the well at any time t.

b) At what rate is the amount of oil in the well decreasing when there are 600,000 gallons of oil remaining?

c) In order not to lose money at what time t should oil no longer be pumped from the well?

14) The rate of consumption of cola in the United States is given by S(t) = Ce^{kt}, where s is measured in billions of gallons per year and t is measured in years from the beginning of 1980.

a) The consumption rate doubles every 5 years and the consumption rate at the beginning of 1980 was 6 billion gallons per year. Find C and k.

b) Find the average rate of consumption of cola over the 10-year time period beginning January 1, 1983. Indicate units of measure.

c) Use the trapezoidal rule with four equal subdivisions to estimate $\int_5^7 S(t)dt$

d) Using correct units, explain the meaning of $\int_5^7 S(t)dt$ in terms of cola consumption.

15) Consider the differential equation $\frac{dy}{dx} = \frac{xy}{x^2+4}$ $\int \frac{1}{y}\,dy = \frac{1}{2}\int \frac{2x}{x^2+4}\,dx$ $u = x^2+4$ $du = 2x\,dx$

a) Find a general solution to the differential equation. $\ln y = \frac{1}{2}\ln(x^2+4) + C$

$y = Ce^{\ln(x^2+4)^{1/2}} = \boxed{C\sqrt{x^2+4}}$ $e^{\ln y} = e^{\frac{1}{2}\ln(x^2+4)} \cdot e^{C}$

b) Find the particular solution that satisfies the initial condition y(0) = 4.

$\ln 4 = \frac{1}{2}\ln 4 + C$ $\ln y = \frac{1}{2}\ln(x^2+4) + \ln 2$ $\boxed{y = 2\sqrt{x^2+4}}$

$C = \ln 4 - \ln 2 = \ln(\frac{4}{2}) = \ln 2$ $e^{\ln y} = e^{\ln(x^2+4)^{1/2}} \cdot e^{\ln 2}$

c) Find the domain and range of the function found in part b.

D: $(-\infty, \infty)$ R: $[0, \infty)$

16) The function f is differentiable for all real numbers. The point $\left(3, \frac{1}{4}\right)$ is on the graph of $y = f(x)$, and the slope at each point (x, y) on the graph is given by $\frac{dy}{dx} = y^2(6-2x)$.

a) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $\left(3, \frac{1}{4}\right)$.

$\frac{d^2y}{dx^2} = y^2(-2) + (6-2x)\cdot 2y\frac{dy}{dx}$

$= -2y^2 + (6-2x)(2y)(y^2)(6-2x)$

$= -2y^2 + 2y^3(6-2x)^2\Big|_{(3,1/4)} = \boxed{-\frac{1}{8}}$

b) Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = y^2(6-2x)$ with the initial condition $f(3) = \frac{1}{4}$.

$\int \frac{1}{y^2}\,dy = \int (6-2x)\,dx$

$-\frac{1}{y} = 6x - x^2 + C$ $-\frac{1}{y} = 6x - x^2 - 13$

$-4 = 18 - 9 + C$ $\frac{1}{y} = x^2 - 6x + 13$

$C = -13$ $\boxed{y = \frac{1}{x^2-6x+13}}$

What can we do with data?

How can we approximate a derivative? using slope equation

How can we approximate a definite integral? Riemann Sums or Trapezoidal Sum

The rate of consumption of cola in the US, S(t), is given in the table below, where S is measured in billions of gallons per year and t is measured in years from the beginning of 1980.

Time (from 1980)	0	2	4	6	8	10	12	14	16
S(t) Billions of gallons per year	6	7.9	10.4	13.8	18.2	24	31.7	41.8	55.1

a) $\frac{S(10)-S(0)}{10-0} = \frac{24-6}{10} = \frac{18}{10}$

a. Using the data in the table, determine the average rate of increase in the rate of soda consumption for the ten year period beginning in 1980. $\boxed{\frac{9}{2}}$ billions of gallons/yr

b. Using the data in the table, approximate the rate of increase in the rate of soda consumption at t = 10. $S'(10) = \frac{S(12)-S(8)}{4} = \frac{31.7-18.2}{4} = \boxed{\frac{27}{8}}$ billions of gals/yr

c. Using the data in the table and a midpoint Riemann sum with 4 subintervals, approximate the value of the definite integral $\int_0^{16} S(t)\,dt$. see margin

d. Using correct units, explain the meaning of your answer in c. see margin

e. A student proposes that the equation $C(t) = 6e^{(.139t)}$ models the data well. Find the value of $C'(10)$ and explain its meaning in the context of the problem. $C'(10) = 3.348$; In 1990, the US was consuming cola at a rate of 3.348 billions of gallons per year

f. Write the equation of a line tangent to $C(t)$ at t = 10 and use it to approximate $C(11)$. $y - 24.09 = 3.348(x-10)$; $C(11) \approx 3.348(11-10) + 24.09 = \boxed{27.44}$

g. By comparing your tangent line approximation to the actual value of $C(11)$, what can you conclude about $C''(11)$? $C''(11)$ is positive; $S(t)$ is conc ↑ @ $t=11$

h. Using the student's equation find the value of the definite integral $\int_0^{16} C(t)\,dt$. $\int_0^{16} 6e^{.139t} = 6\int_0^{16} e^{.139t} = \frac{6}{.139} e^{.139t}\Big]_0^{16} = \boxed{355.87}$

i. Using the student's equation find the average rate of consumption of cola over the 10-year period beginning January 1, 1984. Indicate units of measure. $\frac{C(14)-C(4)}{14-4} = \boxed{3.154 \text{ billion galls/year}}$

c) $\int_0^{16} S(t)\,dt \approx 4(7.9+13.8+24+41.8) = \boxed{350}$

d) From 1980 to 1996, the US consumed approx 350 billion gallons of cola

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work for e) $6e^{.139t}(.139)\Big|_{t=10} = 3.348$

TJH

MVT: (for integrals)

Average Value $\frac{1}{b-a}\int_a^b f(x)\,dx$

Part I Find the average value of the function f on the given interval.

1. $f(x) = x^2$ on [-1,1] $\frac{1}{2}\cdot\frac{x^3}{3}\Big]_{-1}^{1} = \boxed{\frac{1}{3}}$

2. $f(x) = x^3 - 3x^2$ on [-2,1] $\frac{1}{3}\cdot\left[\frac{x^4}{4} - x^3\right]_{-2}^{1} = \boxed{-\frac{17}{4}}$

3. $f(x) = 2\sin x - \cos x$ on $[0, \frac{\pi}{2}]$ $\frac{1}{\pi/2}\cdot\left[-2\cos x - \sin x\right]_0^{\pi/2} = \boxed{\frac{2}{\pi}}$

4. $f(x) = \sqrt{4-x}$ on [0,4] $\frac{1}{4}\cdot\left[-\frac{2}{3}(4-x)^{3/2}\right]_0^4 = \boxed{\frac{4}{3}}$

Part II Find the c guaranteed by the mean value theorem for integrals.

5. $f(x) = 4x^3$ on [1,2]

$4c^3 = \frac{1}{2-1}\int_1^2 4x^3\,dx$

$4c^3 = 15 \quad c = \sqrt[3]{15/4} \approx \boxed{1.554}$

6. $f(x) = x^2 + 4x + 1$ on [0,2]

$c^2+4c+1 = \frac{1}{2}\int_0^2 (x^2+4x+1)\,dx$

$c^2+4c+1 = 19/3$

$c^2+4c-\frac{16}{3} = 0$

$\boxed{c \approx 1.055} \quad \cancel{c \approx 5.055}$

AP PROBLEM 2004

1. Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled byte function F defined by

$$F(t) = 82 + 4\sin\frac{t}{2} \text{ for } 0 \le t \le 30$$

a) To the nearest whole number, how many cars pass through the intersection over the 30 minute period? $\int_0^{30}\left[82+4\sin\left(\frac{t}{2}\right)\right]dt = \left[82t - 8\cos\left(\frac{t}{2}\right)\right]_0^{30} = (2460 - 8\cos 15) - (-8\cos(0))$

$\boxed{= 2474 \text{ cars}}$

b) Is the traffic flow increasing or decreasing at t = 7? Give a reason for your answer.

$F'(7) = 4\cos\left(\frac{t}{2}\right)\cdot\frac{1}{2}\Big|_{t=7} = 2\cos(7/2) = -1.873$. Traffic flow is decreasing at t=7 b/c $F'(7)<0$.

c) What is the average value of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure. $\frac{1}{5}\int_{10}^{15}\left(82+4\sin\frac{t}{2}\right)dt = \frac{1}{5}\left[82t - 8\cos\left(\frac{t}{2}\right)\right]_{10}^{15} = \frac{1}{5}[409.496] = \boxed{81.899 \text{ cars/min}}$

d) What is the average rate of change of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure. $\frac{F(15)-F(10)}{5} = \boxed{1.518 \text{ cars/min}^2}$

AVERAGE VALUE AND AVERAGE RATE OF CHANGE

Find the average value of the function and the average rate of change of the function.

	Avg value	Avg Rate
1. $f(x) = x^2 - x + 1$ on [-1,2]	$\frac{1}{3}\int_{-1}^{2}(x^2-x+1)\,dx = \frac{1}{3}\left[x^3/3 - x^2/2 + x\right]_{-1}^{2} = \frac{1}{3}\left(\frac{9}{2}\right) = \boxed{\frac{3}{2}}$	$\frac{f(2)-f(-1)}{3} = \frac{0}{3} = \boxed{0}$
2. $f(x) = x^3 - 3x^2$ on [-2,1]	$\frac{1}{3}\int_{-2}^{1}(x^3-3x^2)\,dx = \frac{1}{3}\left[\frac{x^4}{4} - x^3\right]_{-2}^{1} = \frac{1}{3}\left(-\frac{51}{4}\right) = \boxed{-\frac{17}{4}}$	$\frac{f(1)-f(-2)}{3} = \frac{-2+20}{3} = \boxed{6}$
3. $f(x) = \sin x$ on $[0, \pi/4]$	$\frac{1}{\pi/4}\int_0^{\pi/4}\sin x\,dx = \frac{4}{\pi}\left[-\cos x\right]_0^{\pi/4} = \frac{4}{\pi}\left(-\frac{\sqrt{2}}{2}+1\right) = \frac{4}{\pi}\left(\frac{-\sqrt{2}+2}{2}\right) = \boxed{\frac{-2\sqrt{2}+4}{\pi}}$	$\frac{f(\pi/4)-f(0)}{\pi/4} = \frac{\frac{\sqrt{2}}{2}-0}{\pi/4} = \boxed{\frac{2\sqrt{2}}{\pi}}$

Particle Motion

If $x(t)$ represents the position of a particle along the x-axis at any time t, then the following statements are true.

1) "Initially means when______________ = 0.
2) "At the origin" means____________ = 0
3) "At rest" means____________=0
4) If the velocity of the particle is positive, then the particle is moving to the____________.
5) If the velocity of the particle is___________, then the particle is moving to the left.
6) To find average velocity over a time interval, divide the change in __________by the change in time.
7) Instantaneous velocity is the velocity at a single moment in time.
8) If the acceleration of the particle is positive, then the____________is increasing.
9) If the acceleration of the particle is ___________, then the velocity is decreasing.
10) In order for a particle of change directions, the__________must change signs.

WHAT YOU NEED TO KNOW.

1) Speed is the absolute value of_____________.
2) If the velocity and acceleration have the same sign, the speed is____________
3) If velocity and acceleration are___________in sign (one positive and the other negative), then speed is decreasing.

<u>There are three ways to use an integral in the study of motion that are easily confused.</u>

4) $\int v(t)dt$ is an _____________integral. It will give you an expression for _____________at time t. Don't forget that you will have a _______, the value of which can be determined if you know a position value at a particular time.

5) $\int_{t_1}^{t_2} v(t)dt$ is a ____________integral and so the answer will be a___________. The number represents the change in ________over the time interval. By the Fundamental Theorem of Calculus, since $v(t) = x'(t)$, the integral will yield $x(t_2) - x(t_1)$. This is known as displacement. The answer can be positive or ________ depending upon if the particle lands to the ______or left of its original starting position.

6) $\int_{t_1}^{t_2} |v(t)|\,dt$ is also a __________integral and so the answer will be a number. The number represents the ______________traveled by the particle over the time interval. The answer should always be__________.

1) The data in the table below give selected values for the velocity, in meters /minute, of a particle moving along the x-axis. The velocity v is a differentiable function of time t.

Time t (min)	0	2	5	6	8	12
Velocity v(t) m/m	-3	2	3	5	7	5

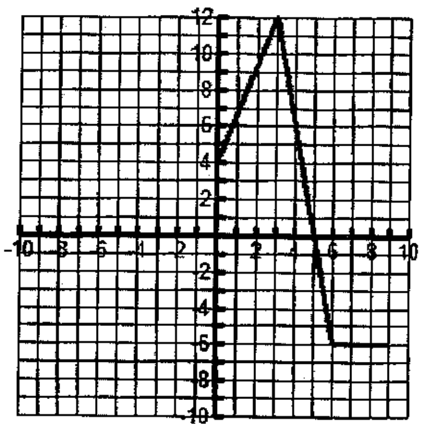
a) At t = 0, is the particle moving to the right or to the left? Explain your answer.

b) Is there a time during the time interval $0 \le t \le 12$ minutes when the particle is at rest? Explain your answer.

c) Use data from the table to find an approximation for $v'(10)$ and explain the meaning of $v'(10)$ in terms of the motion of the particle. Show the computations that lead to your answer and indicate units of measure.

d) Let a(t) denote the acceleration of the particle at time t. Is there guaranteed to be a time t = c in the interval $0 \le t \le 12$ such that a(c) = 0? Justify your answer.

2) The graph at the right represents the velocity v, in feet per second of a particle moving along the x-axis over the time interval from t = 0 to t = 9 seconds.



a) At t = 4 seconds, the particle moving to the right or left? Explain.

b) Over what time interval is the particle moving to the left. Explain.

c) At t = 4 seconds, the acceleration of the particle positive or negative? Explain.

d) What is the average acceleration of the particle over the interval $2 \le t \le 4$. Show the computations that lead to your answer and indicate units of measure.

e) Is there guaranteed to be a time t in the interval $2 \le t \le 4$ such that $v'(t) = -3/2$? Justify your answer.

f) At what time t in the given interval is the particle farthest to the right. Explain.

3) A particle moves along the x-axis so that at time t its position is given by $x(t) = t^3 - 6t^2 + 9t + 11$

a) At t = o, is the particle moving to the right or to the left. Explain.
b) At t = 1 is the velocity of the particle increasing or decreasing? Explain.
c) find all values of t for which the particle is moving to the left.
d) Find the total distance traveled by the particle over the time interval $0 \le t \le 5$

Area/Volume

2002 AB1 and BC1

Let f and g be the functions given by $f(x) = e^x$ and $g(x) = \ln x$

a) Find the area of the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$

b) Find the volume of the solid generated when the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$ is revolved about the line $y = 4$.

c) Let h be the function given by $h(x) = f(x) - g(x)$. Find the absolute minimum value and absolute maximum value of $h(x)$ on the closed interval [1/2, 1]. Justify your answer.

2002 BC3 Form B

Let R be the region in the first quadrant bounded by the y-axis and the graphs of $y = 4x - x^3 + 1$ and $y = 3/4x$.

a) find the area of R.

b) Find the volume of the solid generated when R is revolved about the x-axis.

1997 BC3

Let R be the region enclosed by the graphs of $y=\ln(x^2 + 1)$ and $y = \cos x$.

a) Find the area of R.

b) The base of a solid is the region R. Each cross section of the solid perpendicular to the x-axis is an equilateral triangle. Write an expression involving one or more integrals that gives the volume of the solid. Do not evaluate.

Important Last Minute Review

Work these on **notebook paper**. Use your calculator on problem 1.

1. The population $P(t)$ of fish in a lake satisfies the differential equation $\frac{dP}{dt} = P^2$

   (a) If $P(0)=4000$, what is $\lim_{t\to\infty} P(t)$? Is the solution curve increasing or decreasing? Justify your answer.

   (b) If $P(0)=4000$, what is the population when t = 50? Is the rate of change speeding up or slowing down? Justify your answer.

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2. $\int x\sin(2x^2)\,dx =$

3. $\int \frac{dx}{x^2-6x+10} =$

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4. Given the curves y = cos x and y = 0, write an integral expression which gives the area of the 1st region in Quadrant 1.

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5. Given $\frac{dy}{dx} = \frac{xy}{2}$. Let $f(x)$ be the particular solution to the given differential equation with initial condition $f(0)=3$. Find the solution to this differential equation. Find the value of the slope of the tangent when x = 0

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6. Given $v(t)=3t^2-4$ At time t = 2, the object is at position (4, 0).

   (a) Find the speed and the acceleration at time t = 2.

   (b) Find the total distance traveled by the object over the time interval $1\le t\le 4$.

   (c) Find the position of the object at time t = 3.

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7. If $y = xy + x^2 + 1$, then $\frac{dy}{dx} = ?$

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8. $\lim_{h\to 0} \frac{\cos\left(\frac{\pi}{2}+h\right)-\cos\left(\frac{\pi}{2}\right)}{h} =$

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9. The function f has derivatives of all orders for all real numbers x. Assume that $f(2)=5$,

   $f'(2)=-3$, $f''(2)=4$, $\int_2^5 f'(x)dx = 9$

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(a) Write the equation of the line tangent to f(x) at x = 2

(b) Use your answer to (a) to approximate $f(1.9)$.

(c) Calculate f(5)

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10. $f(x)=\begin{cases} 2x+b & x<3 \\ x^2+ax & x\geq 3 \end{cases}$ Solve for a and b so that f(x) is differentiable at x = 3.
